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ON CLONING CONTEXT-FREENESS

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1. Introduction

A first-order language is called monadic if its non-logical vocabulary lacks both function symbols and relation symbols of arity greater than one. Formulas of this kind of language can be mechanically tested for validity but its expressive power seems to be too restricted to be of any interest for linguistics. Monadic second-order languages on the other hand extend the language of (full) first-order logic by providing for quantifiable monadic second-order variables. These languages increase the definitional resources of first-order logic substantially but they still retain many meta-mathematical properties which make them a 'good source of expressive and manageable theories.' (Gurevich 1985)

In the paper from which the preceding remark is taken Gurevich gives a detailed account of two strong decidability techniques which were developed in connection with monadic second-order theories. One of these techniques is based on games and automata while the other employs generalized products. The applicability of these methods to certain second-order theories where quantification is restricted to variables ranging over subsets of the domain of discourse underlines the fact that formalizations couched in this linguistic framework are supported by a manageable underlying logic. In sharp contrast with the situation exemplified by monadic first-order logic the decidability of applied forms of its second-order counterpart detracts nothing from its value as a flexible expressive means for the presentation of theories which embody interesting parts of either mathematics or the empirical sciences.

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This is not the proper place to substantiate this last claim as far as mathematical theories are concerned. The interested reader is referred to the paper by Gurevich where she can find a wealth of examples. The development which brought linguistics into the picture was initiated by Rogers (1994) who showed in his dissertation that the majority of technical assumptions which form the core of the Principles and Parameter incarnation of transformational grammar are conducive to transparent formalization in monadic second-order tree logic. Using Rabin's second-order logic with multiple successors he was able to give convincing definitions of such central notions as government, chain and category and, in addition to these positive results, to make a significant contribution to the theory-internal debate among students of the Principles and Parameter model by proving that such notions as free-indexing would make Rabin's logic undecidable when added to its vocabulary. Especially this delimiting consequence constitutes a definite advantage of Roger's approach over competing attempts at formalizing the theoretical concepts underlying different versions of transformational grammar. The main drawback of these attempts consists in an excess of expressive power, which makes it extremely difficult to establish any formal criteria to separate the empirically fruitful from the spurious constructs in the Principles and Parameter model.

Unfortunately, there remains one difficulty which threatens to obstruct the project of using monadic second-order logic as the proper framework for a restrictive linguistic metatheory. This difficulty has its origin in the close relationship between recognizable tree sets and weak monadic second-order logic of multiple successors. It has been known since the pioneering work of Doner (1970) and Thatcher and Wright (1968) that the sets of finite trees which can be generated by regular tree grammars are identifiable with the collections of finite trees definable by formulae of the weak version of Rabin's logic. Since the corresponding string sets, i.e. the concatenation of the labels that decorate the leaves of trees in either of these equivalent classes, are exactly identical to the context-free languages the question of whether monadic second-order logic of multiple successors provides an adequate formal framework for linguistic purposes is inseparable from the issue of the proper location of natural languages on the Chomskyan hierarchy. Evidence from languages like Swiss-German (Shieber 1985) and Bambara (Culy 1985) suggests that natural languages exhibit a certain degree of context-sensitivity and that a formal system which aims at serving as an adequate mathematical basis for the working linguist must accommodate a corresponding amount of structural complexity.

There are serious obstacles, however, to any program devoted to a strengthening of Rabin's logic. As has been shown by several people, even minor additions to the vocabulary of the underlying language lead to undecidability (cf. Läuchli and Savioz 1987). The set of theorems of monadic tree theory, another designation of the theory of multiple successors which refers to its intended model, seems to be of so high a degree of complexity that enriching its language always is in danger of overshooting the mark. A natural alternative to an extension of the basic vocabulary of the language would be a restricted form of second-order polyadic quantification. In such a move one would allow relational variables into the language, together with quantifiers binding them. In sharp contrast to the monadic set variables which can be assigned an arbitrary subset of the universe of discourse as a value, the range of interpretation of these new variables would have to be narrowed to special relations over the universe of discourse. Earlier work on the monadic theory lets one expect that a careful restriction of the quantificational realm along the indicated lines could generate the desired supplementary expressiveness without losing control over the class of theorems. Of particular significance for the current issue are the results by Rabin (1972) and Siefkes (1975) to the effect that restricting quantification to regular subsets leads to a proper subsystem of the logic, whereas restricting quantification to recursive subsets leaves the class of theorems untouched. The kind of relations in the polyadic case one will have to consider should perhaps not be determined with reference to their computational complexity, but rather with reference to their internal structure. The recent characterization of context-free languages as sets of strings which can be defined by first-order sentences with an initial second-order existential quantifier provides an instructive example for such a semantic approach in a related context. Extending the work of Büchi (1960) and Elgot (1961) Lautemann et al. (1995) have shown that the model sets satisfying Σ^1_1 sentences of dyadic second-order logic of one successor coincide with the context-free languages if the existential second-order quantifier ranges over binary relations that display the dependencies of matching parentheses. It is an open problem whether the introduction of similarly restricted dyadic quantifiers into the tree language with multiple successors would be accompanied by a controlled increase of the definitional resources and in this way vindicate the idea of a secondary semantics for the limited amount of context-sensitivity that is exemplified by natural languages.

We cannot end this short outline of possible ways out of the narrow confines of context-freeness without mentioning free variable second-order logic. This logic is expressed in a language in which relation and function variables have free, but no bound occurrences. What this means for our monadic second-order tree language is that set variables may still be syntactically bound by higher-order quantifiers but the free relation and function variables get an interpretation which treats them implicitly as universally quantified. Again, it appears to be an open question whether this extension of syntactic resources steers clear of undecidability.

Even though one of the sketched strengthenings of the underlying tree logic—extension of the non-logical vocabulary, restricted polyadic second-order quantification or the introduction of free function and relation variables—may secure just the right increase in expressive power, the precarious status of the monadic second-order theory of multiple successors verging on the undecidable gives little cause for hope. We shall therefore propose a hybrid solution to the problem of how to account for mildly context-sensitive phenomena with the help of tree logic. The limited expressive power of this logic in its original set-up makes it impossible to formulate the solution in a way that would deal directly with the problematic phenomena, but we can give these phenomena a slightly different appearance whereby they do become context-free and as such definable in tree logic.

To be somewhat more specific, we will use an "algebraicized" variant of grammars with macro-like productions (Fischer 1968) to present an analysis of structural dependencies which are beyond the reach of context-free devices. Macro grammars constitute a natural extension of traditional rewrite systems. Whereas in traditional presentations of rule systems for abstract language families the emphasis has been on a first-order substitution process in which auxiliary variables are replaced by elements of the carrier of the proper algebra—concatenations of terminal symbols and auxiliary category variables in the string case—macro grammars lift this process to the second-order level of operations defined on the elements of the carrier of the algebra. Once this second-order substitution process is turned into a process involving only individual variables we are back on familiar ground, the only difference being the shift from single-sorted to many-sorted algebras.

When, two paragraphs back, we spoke about two different representations of context-sensitive dependencies we had this shift from a functional to an objectual substitution process in mind. What makes the transformation of the higher-order into an objectual substitution possible is the technical device of a derived or lifted alphabet. In informal terms, the formation of a derived alphabet can be explained as a nominalization process turning functions into objects. As a result, these latter saturated entities have to rely on an explicit composition operation that provides the necessary glue for combining the objectual counterpart of the original functional entity with its arguments. Since the objectual counterpart now belongs to the type of saturated entities, the substitution process on the symbolic level, when applied to the nominalized expressions resulting from the derivation of the original alphabet, can then take the familiar form of first-order substitution. As should be obvious from the preceding remarks, the language generated by this first-order substitution procedure is context-free and as such amenable to logical analysis in the framework of monadic second-order logic with multiple successors. There is, however, a negative aspect of the approach using derived alphabets. During the translation of the original alphabet into its nominalized derived counterpart new operators representing functional composition—and, furthermore, new constants standing for projection functions which we have suppressed in our informal account—are introduced that change the configurational shape of the translated expressions. This point can be made more vivid in terms of a notational format that is taken for illustrative purposes from lambda calculus with parallel abstraction. Let $\langle x_1, \ldots x_n \rangle$ denote parallel (n-ary) abstraction and $comp(\cdot, \cdot)$ binary application. The lifted variant of the term $f(a, b)$ then corresponds to the following expression $comp(\langle x_1, x_2 \rangle f(x_1, x_2), p(a, b))$, where p represents pair formation in the derived alphabet. This example has to be taken with a grain of salt because in the parallel lambda calculus both the redex and its converted form are supposed to have the same meaning whereas in the context of formal language theory the simple fact that these expressions are different complex symbols embodies two competing proposals for syntactic analysis.

In the last section of the paper we show how to solve the tension between the definitions for linguistic concepts in terms of structural configurations of the original trees and their counterparts in the lifted context. While the definitions that are informed by the original set-up are extensionally inadequate in the general case—they fail to refer to the context-sensitive dependencies—their lifted counterparts, living in an environment of first-order substitution, can be combined with adequate characterizations of those context-sensitive structures. This combination is made possible by the closure of call by value tree languages under deterministic bottom-up tree transducer mappings (see Engelfriet and Schmidt 1978).

We have not attempted to present a defense of macro grammars or of a theory of structural notions embodied in this particular format. A detailed discussion of the (de-)merits of macrogrammatical analyses of a wide range of syntactic problems is contained in H.-P. Kolb's contribution to the present volume. It is worth emphasizing that our application of tree theory to context-sensitive structures is not intended as a justification for a particular form of syntactic analysis. This task remains to be done and we would be delighted if others investigated the structural restrictions that characterize a program of (derived) syntactic macrotheory.

There are several sources that have influenced the ideas reported here. Apart from the work on the logical characterization of language classes that was mentioned above the development in universal algebra that led to a uniform, signature free treatment of varieties has been our main inspiration. From the very beginning of this development it has been obvious to people working in this field that closed sets of operations are best presented as categories with finite products. When this presentation is retranslated into the language of universal algebra we are confronted with a signature whose only non-constant operators are symbols for target tupling and functional composition. Algebras with signatures of this type will play a major role in the paper and they will provide the technical foundation for extending the logical methods of descriptive complexity theory to context-sensitive phenomena.

The first to see the potential for tree language theory of this type of lifted signature was Maibaum (1974). He showed in particular how to map context-free into regular tree productions rules. Unfortunately, a substantial part of his results are wrong because he mistakenly assumed that for so called call by name derivations an unrestricted derivation in a lifted signature would leave the generated language unchanged. Engelfriet and Schmidt (1977, 1978) point out this mistake and give fixed-point characterizations for both call by name and call by value context-free production systems. We hope that the present paper complements the denotational semantics for call by name tree languages by giving an operational analysis of the derivation process both on the level of the original and on the level of the lifted signature.

2. Preliminaries

The purpose of this section is to fix notations and to present definitions for the basic notions related to universal algebra. The key notion is that of a derived algebra. As was indicated above, the realization that derived algebras allow for a different presentation based on a lifted signature constitutes the main step in the process of restoring context-freeness. In this context, we will be involved in a generalization of formal language concepts. Many-sorted or heterogeneous algebras provide the proper formal environment to express this generalization. For expository purposes we will not give every definition in its most general form but keep to the single-sorted case where the reader can easily construct the extension to a situation where more than one sort matter.

DEFINITION 1. *Let S be a set of sorts (categories). A* ***many–sorted alphabet*** *Σ is an indexed family $\langle \Sigma_{w,s} \mid w \in S^*, s \in S\rangle$ of disjoint sets. A symbol in $\Sigma_{w,s}$ is called an* ***operator*** *of* ***type*** *$\langle w, s\rangle$,* ***arity*** *w,* ***sort*** *s and* ***rank*** *$l(w)$. If $w = \varepsilon$ then $f \in \Sigma_{\varepsilon,s}$ is called a* ***constant*** *of sort s. $l(w)$ denotes the length of w.*

Note that a ranked alphabet in the traditional terminology can be identified with an S–sorted alphabet where $S = \{s\}$. The set $\Sigma_{s^n,s}$ is then the same as Σ_n. In the way of explanation, let us print out that each symbol f in $\Sigma_{w,s}$ represents an operation taking n arguments, the ith argument being of sort w_i, and yielding an element of sort s, where $w = w_1 \cdots w_n$. Alternative designation for many-sorted alphabets are *many-sorted signatures* or *many-sorted operator domains*. We list some familiar examples of single-sorted signatures for further reference.

EXAMPLE 1.

a) $\Sigma_0 = \{\varepsilon\} \cup V \qquad \Sigma_2 = \{\frown\}$
Single–sorted signature of semi–groups, extended by a finite set of constants V.

b) $\Sigma_0 = \{\varepsilon\} \qquad \Sigma_1 = \{a \mid a \in V\}$
Single–sorted signature of a monadic *algebra.*

c) $\Sigma_2 = \{\wedge, \vee\}$
Single–sorted signature of lattices.

As was mentioned above, a full description of the theory of a class of algebras of the same similarity type is given by the totality of the derived operations. These operations can be indicated by suitably constructed terms over the basic operators and a set of variables.

DEFINITION 2. *For a many–sorted alphabet Σ, we denote by $T(\Sigma)$ the family $\langle T(\Sigma, s) \mid s \in S\rangle$ of* ***trees of sort*** *s over Σ. $T(\Sigma, s)$ is inductively defined as follows:*

(i) *For each sort $s \in S$*

$$\Sigma_{\varepsilon,s} \subseteq T(\Sigma, s)$$

(ii) *For* $n \geq 1$ *and* $s \in S$, $w \in S^*$, *if* $f \in \Sigma_{w,s}$ *and for* $1 \leq i \leq n$, $t_i \in T(\Sigma, w_i)$, $l(w) = n$,

$$f(t_1, \ldots, t_n) \in T(\Sigma, s)$$

DEFINITION 3. *For a many–sorted alphabet* Σ *and a family of disjoint sets* $Y = \langle Y_s \mid s \in S \rangle$, *the family* $T(\Sigma, Y)$ *is defined to be* $T(\Sigma(Y))$, *where* $\Sigma(Y)$ *is the many–sorted alphabet with* $\Sigma(Y)_{\varepsilon,s} = \Sigma_{\varepsilon,s} \cup Y_s$ *and for* $w \neq \varepsilon$, $\Sigma(Y)_{w,s} = \Sigma_{w,s}$ *We call a subset* $\mathcal{L}$ *of* $T(\Sigma, s)$ *a* ***tree language*** *over* Σ *(of sort* s*).*

Having described the syntax of the tree terms and having indicated their intended interpretation, it remains to specify the central notion of an algebra and to give a precise definition of the way in which the formal term symbols induce an operation on an algebra.

DEFINITION 4. *Suppose that* S *is a set of sorts and that* Σ *is a many–sorted alphabet. A* Σ*-**algebra*** $\mathfrak{A}$ *consists of an* S*-indexed family of sets* $\mathcal{A} = \langle A_s \rangle_{s \in S}$ *and for each operator* $\sigma \in \Sigma_{w,s}$, *a function* $\sigma_{\mathfrak{A}} : A^w \longrightarrow A^s$ *where* $A^w = A^{w_1} \times \cdots \times A^{w_n}$ *and* $w = w_1 \cdots w_n$.

The family $\mathcal{A}$ *is called the sorted* ***carrier*** *of the algebra* $\mathfrak{A}$ *and is sometimes written* $|\mathfrak{A}|$.

Different algebras, defined over the same operator domain, are related to each other if there exists a mapping between their carriers that is compatible with the basic structural operations.

DEFINITION 5. *A* Σ*-**homomorphism*** *of* Σ*-algebras* $h : \mathfrak{A} \longrightarrow \mathfrak{B}$ *is an indexed family of functions* $h_s : A_s \longrightarrow B_s$, $(s \in S)$ *such that for every operator* σ *of type* $\langle w, s \rangle$

$$h_s(\sigma_{\mathfrak{A}}(a_1, \ldots, a_n)) = \sigma_{\mathfrak{B}}(h_{w_1}(a_1), \ldots, h_{w_n}(a_n))$$

for every n*-tuple* $(a_1, \ldots, a_n) \in A^w$.

The set of sorted trees $T(\Sigma, Y)$ can be made into a Σ-algebra by defining the operations in the following way. For every σ in $\Sigma_{w,s}$, for every $(t_1, \ldots, t_n) \in T(\Sigma, Y)^w$

$$\sigma_{\mathfrak{T}(\Sigma,Y)}(t_1, \ldots, t_n) = \sigma(t_1, \ldots, t_n).$$

If $\mathfrak{A}$ is a Σ-algebra, each term $t \in T(\Sigma(Y_w), s)$ induces a function

$$t_{\mathfrak{A}} : A^w \longrightarrow A^s$$

called a *derived operation*. In the place of Y we have used the set of sorted variables $Y_w := \{y_{i,w_i} \mid 1 \leq i \leq l(w)\}$. The meaning of the derived operation $t_{\mathfrak{A}}$ is defined as follows: for $(a_1, \ldots, a_n) \in A^w$

$$t_{\mathfrak{A}}(a_1, \ldots, a_n) = \hat{a}(t)$$

where $\hat{a} : \mathfrak{T}(\Sigma, Y_w) \longrightarrow \mathfrak{A}$ is the unique homomorphism with $\hat{a}(y_{i,w_i}) = a_1$.

In intuitive terms, the evaluation of a $\Sigma(Y)$-tree t in a given Σ-algebra proceeds as follows. First, one assigns a value $\hat{a}(y_i) \in A$ to every variable in Y. Then the operations of $\mathfrak{A}$ are applied to these elements of A as directed by the structure of t. There is, though, another description of the action performed by the derived operations on an algebra $\mathfrak{A}$. According to this conception the derived operations of an algebra $\mathfrak{A}$ are the mappings one gets from the projections $y_{\mathfrak{A}}$ by iterated composition with the primitive operations $\sigma_{\mathfrak{A}}$ $(\sigma \in \Sigma)$.

Given any Σ-algebra $\mathfrak{A}$ we can describe the process of determining the set of derived operations within the framework of another algebra that is based on a different signature. In this new signature the symbols of the original alphabet Σ are now treated as constants, as are the projections. The only operations of non-zero arity are the composition functions and the functions of target tupling. By *composition of operations* is meant the construction of an operation h of type $\langle w, s\rangle$ from given operations f of type $\langle v, s\rangle$ and g_i of type $\langle w, v_i\rangle$ where $\langle w, s\rangle, \langle v, s\rangle, \langle w, v_i\rangle \in S^* \times S$. The constructed operation h satisfies the rule

$$h(a) = f(g_1(a), \ldots, g_k(a))$$

where $k = |v|$ and $a \in A^w$.

If the operations of *target tupling* are among the basic operations that are denoted by the symbols of the new signature, the type of composition operations can be simplified to $\langle\langle v, s\rangle, \langle w, v\rangle\rangle, \langle w, s\rangle$. Take again the operations g_i with their types as just indicated. By their target tupling is meant the construction of an operation h of type $\langle w, v\rangle$ that satisfies the rule

$$h(a) = (g_1(a), \ldots, g_k(a))$$

where again $a \in A^w$ and the outer parentheses on the right-hand side indicate the ordered k-tuple of values. Having introduced composition and target tupling and having indicated their intended interpretation the only missing ingredient that remains for us to introduce, before we can define the concept behind the title of the paper, in the collection of projection operations.

The *projection operations* on a Cartesian product A^w are the trivial operations $\pi^w_{w_i}$ satisfying

$$\pi^w_{w_i}(a) = a_{w_i}$$

A *closed set of operations* or, more briefly, a *clone* on a family of non-void sets $A = \langle A_s \rangle$ $(s \in S)$ is a set of operations on A that contains the projection operations and is closed under all compositions. The relevance of clones for the purposes of understanding the hidden structure of an algebra and of providing a signature free treatment for universal algebra was first realized by P. Hall. An alternative proposal under the name of algebraic theories and their algebras is due to J. Bénabou and W. Lawvere. We have chosen to follow the example of J. Gallier and of J. Engelfriet and E. M. Schmidt in using the style of presentation familiar from standard universal algebra. Our only departure from this tradition is the explicit inclusion of the operation of target tupling into an official definition of the clone of term operations.

DEFINITION 6. *The* ***clone of term operations*** *of an* Σ*-algebra* $\mathfrak{A}$*, denoted by* $\mathsf{Clo}(A)$*, is the smallest set of operations on the carrier* $\mathcal{A} = \langle A_s \rangle$ *that contains the primitive operations of* $\mathfrak{A}$*, the projections and is closed under composition and target tupling. The set of all operations of type* $\langle w, s \rangle$ *in* $\mathsf{Clo}(A)$ *is denoted by* $\mathsf{Clo}_{\langle w,s \rangle}(A)$*.*

To actually present a clone as an algebra both the set of sorts S and the underlying alphabet Σ have to be adapted to the new situation. The idea to characterize this new situation with the help of a derived alphabet was first used by J. Bénabou and later rediscovered by T. S. E. Maibaum.

DEFINITION 7. *Let* S *be a set of sorts and* $\Sigma = \langle \Sigma_{w,s} \rangle$ *be an* S*-sorted alphabet* $(\langle w, s \rangle \in S^* \times S)$*. The* ***derived*** $(S^* \times S^*)$***-sorted alphabet of*** Σ *denoted by* $D(\Sigma)$*, is defined as follows:*

$$D(\Sigma) = \Sigma \cup \left\{ \begin{array}{l} \pi_i^v \\ (\ )_{w,v} \\ S_{v,w,s} \end{array} \right.$$

where $w \in S^*$*,* $v \in S^+$ *and* $s \in S$*. Each* π_i^v *is a projection operator of type* $\langle \varepsilon, \langle v, v_i \rangle \rangle$*, each* $(\ )_{w,v}$ *is a tupling operator and each* $S_{v,w,s}$ *is a substitution or composition operator of types* $\langle \langle w, v_1 \rangle \cdots \langle w, v_n \rangle, \langle w, v \rangle \rangle$*, respectively* $\langle \langle v, s \rangle, \langle w, v \rangle, \langle w, s \rangle \rangle$ *and each* σ *in* $\Sigma_{w,s}$ *becomes a constant operator in the derived alphabet* $D(\Sigma)$ *of type* $\langle \varepsilon, \langle w, s \rangle \rangle$*.*

EXAMPLE 2. *We have seen above that trees form the sorted carrier of the* Σ*-algebra* $\mathfrak{T}(\Sigma, Y)$*. What is of fundamental importance for the further development is the fact that trees with variables can also be seen as a* $D(\Sigma)$*-algebra. The tree substitution algebra* $\mathfrak{D}\mathfrak{T}(Y)$ *is a* $D(\Sigma)$*-algebra whose carrier of sort* $\langle w, s \rangle$ *is the set of trees* $T(\Sigma, Y_w)_s$*, i.e. the set of* *trees of sort s that may contain variables in Y_w. In order to alleviate our notation we will denote this carrier by $T(w,s)$. Carriers of sort $\langle w,v\rangle$ are v-tuples of carriers of sort $\langle w,v_i\rangle$ and are denoted by $T(w,v)$. Each σ in $\Sigma_{w,s}$ is interpreted as the tree $\sigma(y_{1,w_1},\ldots,y_{n,w_n})$, each π_i^v is interpreted as y_{i,v_i}, each $(\ )_{w,v}(t_1,\ldots,t_n)$ is interpreted as the formation of v-tuples $(t_1,\ldots,t_n)$, where each t_i is an element of $T(w,v_i)$ and $l(v)=k$, and each $S_{v,w,s}$ is interpreted as a composition or substitution of trees. An intuitive description of the composite $S_{v,w,s_{\mathfrak{D}\mathfrak{T}(\Sigma,Y)}}(t,t')$ with $t\in T(v,s)$ and $t'=(t_1,\ldots,t_k)\in T(w,v)$ is this: the composite is the term of type $\langle w,s\rangle$ that is obtained from t by substituting the term t_i of sort v_i for the variable y_{i,v_i} in t. The formal definition of the composition operation relies on the unique homomorphism $\hat{t}':\mathfrak{T}(\Sigma,Y_v)\longrightarrow\mathfrak{T}(\Sigma,Y_w)$ that extends the function $t':Y_v\longrightarrow\bigcup_i T(w,v_i)$ mapping y_{i,v_i} to t_i in $T(w,v_i)$. Then for any $t\in T(v,s)$, we define $S_{v,w,s_{\mathfrak{D}\mathfrak{T}(\Sigma,Y)}}(t,t')$ as the value of $\hat{t}'$ on the term t:*

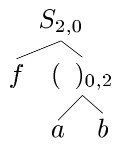
$$S_{v,w,s_{\mathfrak{D}\mathfrak{T}(\Sigma,Y)}}(t,t') := \hat{t}'(t) = t[t_1,\ldots,t_n]$$

where the last term indicates the result of substituting t_i for y_{i,v_i}.

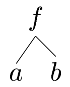
Since the derived alphabet $D(\Sigma)$ leads to a tree algebra $\mathfrak{T}(D(\Sigma))$ in the same way that the alphabet Σ led to the algebra $\mathfrak{T}(\Sigma)$, there is a unique homomorphism $\beta:\mathfrak{T}(D(\Sigma))\longrightarrow\mathfrak{D}\mathfrak{T}(\Sigma)$. It was pointed out by Gallier (1984) that this homomorphism is very similar to β-conversion in the λ-calculus. The explicit specification of its action repeats in a concise form the description of the tree substitution algebra:

$$\begin{array}{rcll}
\beta_{w,s}(\sigma) & = & \sigma(y_{1,w_1},\ldots,y_{n,w_n}) & \textit{for } \sigma\in\Sigma_{w,s}\\
\beta_{w,s}(\pi_i^w) & = & y_{i,w_i} & \textit{if } w_i=s\\
\beta_{w,s}((\ )_{w,v}(t_1,\ldots,t_k) & = & (t_1,\ldots,t_k) & \textit{for } t_i\in T(w,v_i)\\
\beta_{w,s}(S_{v,w,s}(t,t')) & = & \beta_{v,s}(t)[\beta_{w,v_1}(t_1),\ldots,\beta_{w,v_k}(t_k)] & \\
 & & \qquad \textit{for } t\in T(v,s),\ t_i\in T(w,v_i) & \\
 & & \qquad \textit{and } t'=(t_1,\ldots,t_k). &
\end{array}$$

EXAMPLE 3. *Suppose that the set of sorts S is a singleton and that Σ contains three symbols f,a and b where $\Sigma_{ss,s}=\{f\}$ and $\Sigma_{\varepsilon,s}=\{a,b\}$ and s is the single sort in S. As is customary in the context of single-sorted alphabets we shall write the type $\langle s^n,s\rangle$ as n. According to this notational convention the following figure displays a tree t in $T(D(\Sigma))$:*



Applying β_0 to this tree it returns as a value the tree $\beta_0(t) = f(a, b)$ in $T(\Sigma)$. Displayed in tree form this last term looks as follows:



The unique homomorphism β from the derived tree algebra into the tree substitution algebra has a right inverse $\mathsf{LIFT}^\Sigma = (\mathsf{LIFT}^\Sigma_{w,s})(\langle w, s\rangle \in S^* \times S)$, where $\mathsf{LIFT}^\Sigma_{w,s}$ maps a Σ-tree in $T(\Sigma(Y_w), s)$ into a $D(\Sigma)$-tree in $T(D(\Sigma), \langle \varepsilon, \langle w, s\rangle\rangle)$. Since we will have no occasion to apply the function LIFT to trees over a many-sorted alphabet we content ourselves with giving its recursive definitions for the single-sorted case:

Definition 8. *Suppose that Σ is a single-sorted or ranked alphabet. For $k \geq 0$, $\mathsf{LIFT}^\Sigma_k : T(\Sigma, X_k) \longrightarrow T(D(\Sigma), \langle \varepsilon, k\rangle)$ is the function defined recursively as follows (where $X_k = \{x_i \mid 1 \leq i \leq k\}$):*

$$
\begin{array}{rcl}
\mathsf{LIFT}^\Sigma_k(x_i) & = & \pi^k_i \\
\mathsf{LIFT}^\Sigma_k(\sigma) & = & S_{0,k,1}(\sigma) \text{ for } \sigma \in \Sigma_0 \\
\mathsf{LIFT}^\Sigma_k(\sigma(t_1, \ldots, t_n)) & = & S_{n,k,1}(\sigma, (\ )_{k,n}, (\mathsf{LIFT}^\Sigma_k(t_1), \ldots, \mathsf{LIFT}^\Sigma_k(t_n)) \\
 & & \text{for } \sigma \in \Sigma_n.
\end{array}
$$

It should be obvious that for any tree t in $T(\Sigma, X_k)$

$$\beta_k(\mathsf{LIFT}^\Sigma_k(t)) = t.$$

The reader will have noticed that the tupling operator was conspicuous by its absence in the recursive definition of the LIFT-function. According to our official definition for the carrier of the tree substitution algebra a term like $\sigma(t_1, \ldots, t_n)$ is the result of composing the n-tuple $t_1, \ldots, t_n$ of k-ary terms t_i with the n-ary term $\sigma(x_1, \ldots, x_n)$. This part of the "structural history" is suppressed in the third clause of the LIFT-specification. We shall adhere to this policy of eliminating this one layer in the configurational set-up of tree terms and we shall extend this policy to the level of explicit trees over the derived alphabet $D(\Sigma)$. This does not mean that we revoke the official, pedantic definition of the symbol set in a derived alphabet, but we shall make our strategy of notational alleviation type consistent by reading the type of each substitution operator $S_{v,w,s}$ as $\langle\langle v,s\rangle\langle w,v_1\rangle\cdots\langle w,v_n\rangle,\langle w,s\rangle\rangle$ instead of $\langle\langle v,s\rangle\langle w,v\rangle,\langle w,s\rangle\rangle$.

3. Context-Free Tree Languages

The correspondence between trees in explicit form, displaying composition and projection labels, and their converted images as elements of a tree substitution algebra is an example of a situation which is characterized by a meaning preserving relationship between two algebras $\mathfrak{A}$ and $\mathfrak{B}$. Of particular interest to formal language theory is the situation where problems in an algebra $\mathfrak{B}$ can be lifted to the tree level, solved there and, taking advantage of the fact that trees can be regarded as denoting elements in an arbitrary algebra, projected back into their original habitat $\mathfrak{B}$. This transfer of problems to the symbolic level would, of course, produce only notational variants as long as the lifted environment constitutes just an isomorphic copy of the domain in relation to which the problems were first formulated. One might suspect that β-conversion and its right inverse are a case of this type. Despite their suspicious similarity, trees in explicit form and their cousins which are the results of performing the operations according to the instructions suggested by the composition and projection symbols, are sufficiently different to make the transfer of problems to the explicit variants a worthwhile exercise.

In intuitive terms, the difference between the two tree algebras is related to the difference between a first-order and a second-order substitution process in production systems. Let us view grammars as a mechanism in which local transformations on trees can be performed in a precise way. The central ingredient of a grammar is a finite set of productions, where each production is a pair of trees. Such a set of productions determines a binary relation on trees such that two trees t and t' stand in that relation if t' is the result of removing in t an occurrence of a first component in a production pair and replacing it by the second component of the same pair. The simplest type of such a replacement is defined by a production that specifies the substitution of a single-node tree t_0 by another tree t_1. Two trees t and t' satisfy the relation determined by this simple production if the tree t' differs from the tree t in having a subtree t_1 that is rooted at an occurrence of a leaf node t_0 in t. In slightly different terminology, productions of this kind incorporate instructions to rewrite auxiliary variables as a complex symbol that, autonomously, stands for an element of a tree algebra. As long as the carrier of a tree algebra is made of constant tree terms the process whereby nullary variables are replaced by trees is analogous to what happens in string languages when a nonterminal auxiliary symbol is rewritten as a string of terminal and non-terminal symbols, independently of the context in which it occurs. The situation changes dramatically if the carrier of the algebra is made of symbolic counterparts of derived operations and the variables in production rules range over such second-level entities. As we have seen in the preceding sections, the tree substitution algebra provides an example for an algebra with this structure. The following example illustrates the gain in generative power to be expected from production systems determining relations among trees that derive from second-order substitution of operators rather than constants.

EXAMPLE 4. *Let V be a finite vocabulary. It gives rise to a* monadic *signature Σ if all the members of V are assigned rank one and a new symbol ε is added as the single constant of rank zero. For concreteness, let us assume that we are dealing with a vocabulary V that contains the symbols a and b as its only members. Trees over the associated monadic signature $\Sigma = \Sigma_0 \cup \Sigma_1$ where $\Sigma_0 = \{\varepsilon\}$ and $\Sigma_1 = \{a, b\}$ are arbitrary sequences of applications of the operators a and b to the constant ε. It is well known, as was pointed out above, that there is a unique homomorphism from these trees, considered as the carrier of a Σ-algebra to any other algebra of the same similarity type. In particular, there is a homomorphism into V^* when a and b are interpreted as left-concatenation with the symbol a and b, respectively, and when ε is interpreted as the constant string of length zero. This homomorphism establishes a bijection between $T(\Sigma)$ and V^* (cf. Maibaum 1974). When combined with this bijective correspondence the following regular grammar generates the set of all finite strings over V.*

$$G = \langle \Sigma, \mathcal{F}, S, P \rangle$$

$$\begin{array}{lllllll} \Sigma_0 & = & \{\varepsilon\} & \qquad & \Sigma_1 & = & \{a, b\} \\ \mathcal{F}_0 & = & \{S\} & & \mathcal{F}_n & = & \emptyset \text{ for } n \geq 1 \end{array}$$

$$P = \{S \to \varepsilon \mid a(S) \mid b(S)\}$$

$$\mathcal{L}(G, S) = \Sigma_1^*$$

where we have identified $T(\Sigma)$ with Σ_1^*.

P stands for the finite set of productions and s stands for the only non-terminal symbol. V gives also rise to a binary *signature Σ' if the members of V are assigned rank zero and two new symbols are added, ε of rank zero and $\frown$ of rank two. Trees over this signature are nonassociative arc-links between the symbols a and b. When a and b are interpreted as*

constant strings of length one, ε as constant string of length zero and the arc $\frown$ as (associative) concatenation V^* becomes an Σ'-algebra. Note that the unique homomorphism from $\mathfrak{T}(\Sigma')$ to V^* is not *a bijection this time. When combined with this homomorphism the following grammar generates the string language $\{a^n b^n\}$.*

$$G' = \langle \Sigma', \mathcal{F}, S, P' \rangle$$

$$\begin{array}{lcl} \Sigma'_0 & = & \{\varepsilon, a, b\} \qquad\qquad \Sigma'_2 \;=\; \{\frown\} \\ \mathcal{F}_0 & = & \{S\} \end{array}$$

$$P' = \{S \to \varepsilon \mid \frown(a, \frown(S, b))\}$$

$$\mathcal{L}(G', S) = \{\frown(a, \frown(\ldots, \frown(\varepsilon, b) \ldots b) \ldots)\}$$

where n occurrences of a precede the same number of occurrences of b for $n \geq 0$.

Now, consider the $D(\Sigma)$-tree substitution algebra $\mathfrak{D}\mathfrak{T}(\Sigma, X)$. As will be recalled, its carrier consists of trees in $T(\Sigma, X)$. Due to the fact that Σ is a monadic *signature trees in $T(\Sigma, X)$ may not contain more that a single variable. Auxiliary symbols ranging over this carrier take therefore as values monadic or constant derived operators. When monadic auxiliary symbols appear in productions this means that they behave the way that nullary auxiliary symbols do except for the fact that their argument has to be inserted into the unique variable slot of their replacing derived operator. After these explanations it should be obvious that the following grammar over the original* monadic *signature Σ generates the context-free string language $\{a^n b^n\}$ when combined with the unique homomorphism mentioned above:*

$$G'' = \langle \Sigma, \mathcal{F}'', S, P'' \rangle$$

$$\mathcal{F}''_0 \;=\; \{s\} \qquad\qquad \mathcal{F}''_1 \;=\; \{F\}$$

$$P'' = \left\{ \begin{array}{lcl} S & \to & F(\varepsilon) \mid \varepsilon, \\ F(x) & \to & a(F(b(x))) \mid a(b(x)) \end{array} \right\}$$

$$\mathcal{L}(G'', S) = \{\overbrace{a(a \ldots}^{n}\overbrace{(b(b \ldots}^{n}(\varepsilon) \ldots)\}$$

The last grammar in the preceding example illustrates for the language $\{a^n b^n\}$ a transformation that can be applied to the grammar of any given context-free language. Employing terminology to be introduced in a moment, we have the precise characterization of the gain in generative capacity resulting from the introduction of monadic operator variables into production systems: For every finite alphabet V, the context-free subsets of $T(\Sigma_V)$ (where Σ_V is the monadic signature induced by V) have exactly the context-free string languages as values of the unique homomorphism that maps trees in $T(\Sigma_V)$ into strings in V^* and interprets the elements of V as unary operations that concatenate for the left with their argument.

We now turn to introducing the notion of a context-free tree grammar. This type of grammar is related to the type of grammars that were defined by Fischer (1968) and were called macro grammars by him. Context-free tree grammars constitute an algebraic generalization of macro grammars since the use of macro-like productions served the purpose of making simultaneous string copying a primitive operation.

Definition 9. *A* ***context–free tree grammar*** $G = \langle \Sigma, \mathcal{F}, S, P\rangle$ *is a 4-tuple, where* Σ *is a finite ranked alphabet of* terminals*,* $\mathcal{F}$ *is a finite ranked alphabet of* nonterminals *disjoint from* Σ*,* $S \in \mathcal{F}$ *is the start symbol of rank* 0 *and* P *is a finite set of rules of the form*

$$F(x_1, \ldots, x_m) \rightarrow t \qquad (n \geq 0)$$

where $F \in \mathcal{F}_m$ *and* $t \in T(\Sigma \cup \mathcal{F}, X_m)$*. Recall that* X *is here assumed to be a set of variables* $X = \{x_1, x_2, \ldots\}$ *and* $X_m = \{x_1, \ldots, x_m\}$*.*

For reasons having to do with the impossibility of mirroring the process of copying in a grammar with a completely uncontrolled derivation regime we restrict ourselves to one particular mode of derivation. According to this mode a function symbol may be replaced only if all its arguments are trees over the terminal alphabet. In the conventional case this form of replacement mechanism would correspond to a rightmost derivation. In terms of modes of computations, the exclusion of nonterminals that have any nonterminals appearing in any of their arguments corresponds to a *call by value* computation where the actual parameters of a function call have to be values from the domain of computation.

Definition 10. *Let* $G = \langle \Sigma, \mathcal{F}, S, P\rangle$ *be a context-free tree grammar and let* $t, t' \in T(\Sigma \cup \mathcal{F})$*.* t' *is* ***directly derivable*** *by an* ***inside-out step*** *from* t $(t \Rightarrow t')$ *if there is a tree* $t_0 \in T(\Sigma \cup \mathcal{F}, X_1)$ *containing exactly* one *occurrence of* x_1*, a corresponding rule* $F(x_1, \ldots, x_m) \rightarrow t''$*, and trees* $t_1, \ldots, t_m \in T(\Sigma)$ *such that*

$$\begin{array}{rcl} t & = & t_0[F(t_1, \ldots, t_m)] \\ t' & = & t_0[t''[t_1, \ldots, t_m]] \end{array}$$

t' *is obtained from* t *by replacing an occurrence of a subtree* $F(t_1, \ldots, t_m)$ *by the tree* $t''[t_1, \ldots, t_m]$*. By the inside-out restriction on the derivation scheme it is required that the trees* t_1, t_2 *through* t_n *be terminal trees.*

Recall from the preceding section that for $m, n \geq 0$, $t \in T(\Sigma, X_m)$ *and* $t_1, \ldots, t_m \in T(\Sigma, X_n)$ $t[t_1, \ldots, t_m]$ *denotes the result of* ***substituting*** t_i *for* x_i *in* t. *Observe that* $t[t_1, \ldots, t_m]$ *is in* $T(\Sigma, X_n)$.

As is customary $\stackrel{*}{\Rightarrow}$ *denotes the transitive-reflexive closure of* $\Rightarrow$.

DEFINITION 11. *Suppose* $G = \langle \Sigma, \mathcal{F}, S, P \rangle$ *is a context-free tree grammar. We call*

$$\mathcal{L}(G, S) = \{t \in T(\Sigma) \mid S \stackrel{*}{\Rightarrow} t\}$$

the ***context-free inside-out tree language*** *generated by* G *from* S.

We reserve a special definition for the case where $\mathcal{F}$ contains only function symbols of rank zero.

DEFINITION 12. *A* ***regular tree grammar*** *is a tuple* $G = \langle \Sigma, \mathcal{F}, S, P \rangle$, *where* Σ *is a finite ranked alphabet of terminals,* $\mathcal{F}$ *is a finite alphabet of function or nonterminal symbols of rank zero,* $S \in \mathcal{F}$ *is the start symbol and* $P \subseteq \mathcal{F} \times T(\Sigma \cup \mathcal{F})$ *is a finite set of productions. The* ***regular tree language*** *generated by* G *is*

$$\mathcal{L} = \{t \in T(\Sigma) \mid S \stackrel{*}{\Rightarrow} t\}$$

Note that in the case of regular grammars the analogy with the conventional string theory goes through. There is an equivalence of the unrestricted, the rightmost and the leftmost derivation modes where the terms 'rightmost' and 'leftmost' are to be understood with respect to the linear order of the leaves forming the frontier of a tree in a derivation step.

Very early in the development of (regular) tree grammars it was realized that there exists a close relationship between the families of trees generated by tree grammars and the family of context-free string languages. This fundamental fact is best described by looking at it from the perspective on trees that views them as symbolic representations of values in arbitrary domains. Recall the unique homomorphism from the introductory example of this section that mapped non-associative concatenation terms into strings of their nullary constituents. This homomorphism is a particular case of a mapping than can easily be specified for an arbitrary signature.

DEFINITION 13. *Suppose* Σ *is multi-sorted or ranked alphabet. We call* ***yield*** *or* ***frontier*** *the unique homomorphism* y *that interprets every operator in* Σ_w *or* Σ_n *with* $l(w) = n$ *as the* n*-ary operation of concatenation.*

More precisely

$$\begin{array}{rcll} y(\sigma) & = & \sigma & \textit{for } \sigma \in \Sigma_\varepsilon \textit{ (or } \Sigma_0\textit{)} \\ y(\sigma(t_1,\ldots,t_n)) & = & y(t_1)\ldots y(t_n) & \textit{for } \sigma \in \Sigma_w \textit{ (or } \Sigma_n\textit{) and} \\ & & & t_i \in T(\Sigma)_{w_i} \textit{ (or } T(\Sigma)\textit{)} \end{array}$$

Fact *A (string) language is context-free iff it is the yield of a regular tree language.*

As was shown in the introductory example, the addition of macro operator variables increases the generative power of context-free tree grammars over monadic alphabets considerably. The following example demonstrates that the addition of n-ary macro operator variables leads to a significant extension with respect to arbitrary ranked alphabets. The string language of the following context-free tree language is not context-free.

EXAMPLE 5. *Let us consider a context-free tree grammar* $G = \langle \Sigma, \mathcal{F}, S, P\rangle$ *such that its frontier is the set of all cross-dependencies between the symbols a,c and b, d, respectively. The grammar* G *consists of the components as shown below:*

$$\begin{array}{rclcrcl} \Sigma_0 & = & \{\varepsilon, a, b, c, d\} & & \Sigma_2 & = & \{\frown\} \\ \mathcal{F}_0 & = & \{S\} & & \mathcal{F}_4 & = & \{F\} \end{array}$$

$$P = \left\{ \begin{array}{rcl} S & \to & F(a,\varepsilon,c,\varepsilon) \mid F(\varepsilon,b,\varepsilon,d) \mid \varepsilon, \\ F(x_1,x_2,x_3,x_4) & \to & \\ & & F(\frown(a,x_1),x_2,\frown(c,x_3),x_4) \mid \\ & & F(x_1,\frown(b,x_2),x_3,\frown(d,x_4)) \mid \\ & & \frown(\frown(\frown(x_1,x_2),x_3),x_4) \end{array} \right\}$$

$$\mathcal{L}(G,S) =$$

$$\{\frown(\frown(\frown(\frown(a,\ldots)\ldots),\frown(b,\ldots)\ldots),\frown(c,\ldots)\ldots),\frown(d,\ldots)\ldots))\}$$

The number of occurrences of a's and c's and of b's and d's, respectively, has to be the same. By taking the frontier of the tree terms, we get the language $\mathcal{L}' = \{a^n b^m c^n d^m\}$.

The language of the preceding example illustrates a structure that can actually be shown to exist in natural language. Take the following sentences which we have taken from Shieber's (1985) paper:

EXAMPLE 6.

(i) a) Jan säit das mer em Hans es huus hälfed aastriiche.
b) *John said that we helped Hans (to) paint the house.*

(ii) a) Jan säit das mer d'chind em Hans es huus lönd hälfed aastriiche.
b) *John said that we let the children help Hans paint the house.*

The NP's and the V's of which the NP's are objects occur in cross-serial order. *D'chind* is the object of *lönd*, *em Hans* is the object of *hälfe*, and *es huus* is the object of *aastriiche.* Furthermore the verbs mark their objects for case: *hälfe* requires dative case, while *lönd* and *aastriiche* require the accusative. It appears that there are no limits on the length of such constructions in grammatical sentences of Swiss German. This fact alone would not suffice to prove that Swiss German is not a context-free string language. It could still be the case that Swiss German *in toto* is context-free even though it subsumes an isolable context-sensitive fragment. Relying on the closure of context-free languages under intersection with regular languages Shieber is able to show that not only the fragment exhibiting the cross-dependencies but the whole of Swiss German has to be assumed as non context-free.

Shieber intersects Swiss German with the regular language given in Example 7 in (iiia) to obtain the result in (iv). As is well known, this language is not context-free.

EXAMPLE 7.

(iii) a) Jan säit das mer (d'chind)* (em Hans)* händ wele (laa)* (hälfe)* aastriiche.
b) *John said that we (the children)* (Hans)* the house wanted to (let)* (help)* paint.*

(iv) Jan säit das mer (d'chind)^n (em Hans)^m händ wele (laa)^n (hälfe)^m aastriiche.

Swiss German is not an isolated case that one could try to sidestep and to classify as methodologically insignificant. During the last 15 years a core of structural phenomena has been found in genetically and typologically unrelated languages that leaves no alternative to reverting to grammatical formalisms whose generative power exceeds that of context-free grammars.

It has to be admitted that the use of macro-like productions is not the only device that has been employed for the purpose of providing grammar formalisms with a controlled increase of generative capacity. Alternative systems that were developed for the same purpose are e.g. tree adjoining grammars, head grammars and linear indexed grammars. Although these systems make highly restrictive claims about natural language structure their predictive power is closely tied to the individual strategy they exploit to extend the context-free paradigm. The great advantage of the tree oriented formalism derives from its connection with *descriptive complexity theory*. Tree properties can be classified according to the complexity of logical formulas expressing them. This leads to the most perspicuous and fully *grammar independent* characterization of tree families by monadic second-order logic. Although this characterization encompasses only regular tree sets the lifting process of the preceding section allows us to simulate the effect of macro-like productions with regular rewrite rules.

Again, the device of lifting an alphabet into its derived form is not without its alternatives in terms of which a regular tree set can be created that has as value the intended set of tree structures over the original alphabet. Our own reason for resting with the lifting process was the need to carry through the "regularizing" interpretation not only for the generated language, but also for the derivation steps.

A very simple example is now given of a context-free tree grammar that specifies as its frontier the (non-context-free) string language $\{a^n b^n c^n\}$. At the same time we shall present the lifted version of the grammar and illustrate the effect of the productions with two sequences of derivation trees.

EXAMPLE 8. *Consider the context-free tree grammar $G = \langle \Sigma, \mathcal{F}, S, P \rangle$ which consists of the components as shown below:*

$$\begin{array}{rclcrcl} \Sigma_0 & = & \{a, b, c\} & \qquad & \Sigma_2 & = & \{\frown\} \\ \mathcal{F}_0 & = & \{S\} & & \mathcal{F}_3 & = & \{F\} \end{array}$$

$$P = \left\{ \begin{array}{rcl} S & \rightarrow & F(a, b, c) \\ F(x_1, x_2, x_3) & \rightarrow & \\ & & F(\frown(a, x_1), \frown(b, x_2), \frown(c, x_3)) \mid \\ & & \frown(\frown(x_1, x_2), x_3)) \end{array} \right\}$$

Applying the S-production once and the first F-production two times we arrive at the sequence of trees in Figure 1.

The result of applying the terminal F-production to the last three trees in Figure 1 is shown in Figure 2.

Transforming the grammar G with the help of the LIFT *mapping of Definition 8 into its derived correspondent G_D produces* a regular grammar. *As will be recalled from the remarks after Definition 7, all symbols from the original alphabet become constant operators in the derived alphabet. In the presentation below the coding of type symbols is taken over from the*

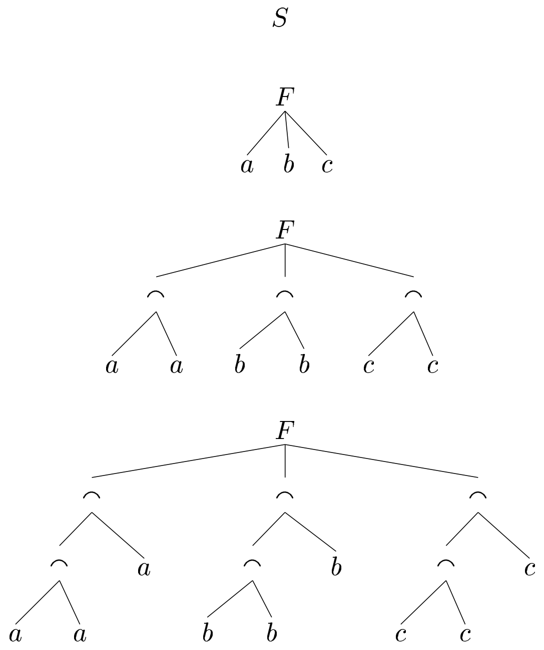


FIGURE 1. An Example for $\{a^n b^n c^n\}$

last example in section 2. It relies upon the bijection between $S^* \times S$ and $\mathbb{N}$, where S is a singleton. Let $\mathbb{N}$ be the set of sorts and let $D(\Sigma)$ be the derived alphabet. The derived grammar $G_D = \langle D(\Sigma), D(\mathcal{F}), D(S), D(P)\rangle$ contains the following components:

$$\begin{array}{lcllcl} D(\Sigma)_0 & = & \{a, b, c\} & D(\Sigma)_{n,k} & = & \{S\} \\ D(\Sigma)_2 & = & \{\frown\} & D(\Sigma)_n & = & \{\pi\} \\ & & & & & \\ \mathcal{F}_0 & = & \{S\} & \mathcal{F}_3 & = & \{F\} \end{array}$$

$$P = \left\{ \begin{array}{lcl} S & \to & S(F, a, b, c) \\ F & \to & S(\frown, S(\frown, \pi_1, \pi_2), \pi_3) \mid \\ \multicolumn{3}{l}{\quad S(F, S(\frown, \pi_1, S(a)), S(\frown, \pi_2, S(b)), S(\frown, \pi_3, S(c)))} \end{array} \right\}$$

The context will always distinguish occurrences of the start symbol S from occurrences of the substitution operator S.

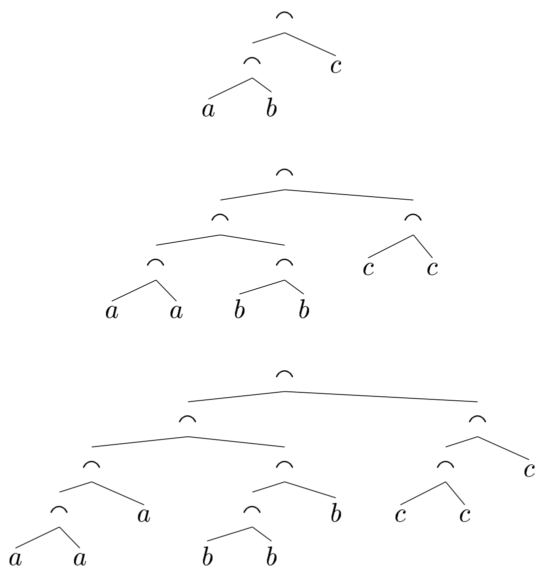


FIGURE 2. An Example of $\{a^n b^n c^n\}$

Sample derivations and two specimens of the generated language $\mathcal{L}(G_D)$ appear in Figures 3 and 4.

The case illustrated by this example is characteristic of the general situation. An arbitrary context-free tree grammar G can be mapped into its derived counterpart G_D with the help of the LIFT transformation. The result of this transformation process, G_D, is a regular grammar and therefore specifies a context-free language as the yield of its generated tree language $\mathcal{L}(G_D)$. This follows directly from the fundamental fact, stated above, that a string language is context-free if and only if it is the leaf or frontier language of a regular tree language. The frontiers of $\mathcal{L}(G)$ and of $\mathcal{L}(G_D)$ are obviously not the same languages. The yield of $\mathcal{L}(G_D)$ in particular consists of strings over the whole alphabet Σ extended by the set of projection symbols. Due to the fact, however, that the composition of the LIFT operation with the β operation is the identity on the elements of $T(\Sigma, X)$, it is of considerable interest to know whether this close relationship between elements of $T(D(\Sigma)$ and of $T(\Sigma, X)$ is preserved by the derivation process in the context-free grammar and its

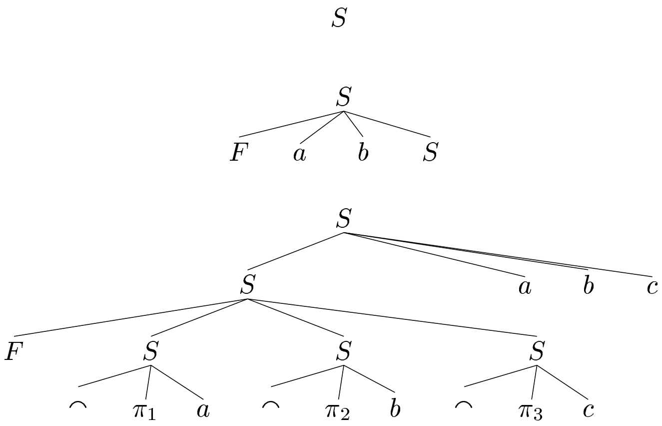


FIGURE 3. Lifted Derivations corresponding to Figure 1.

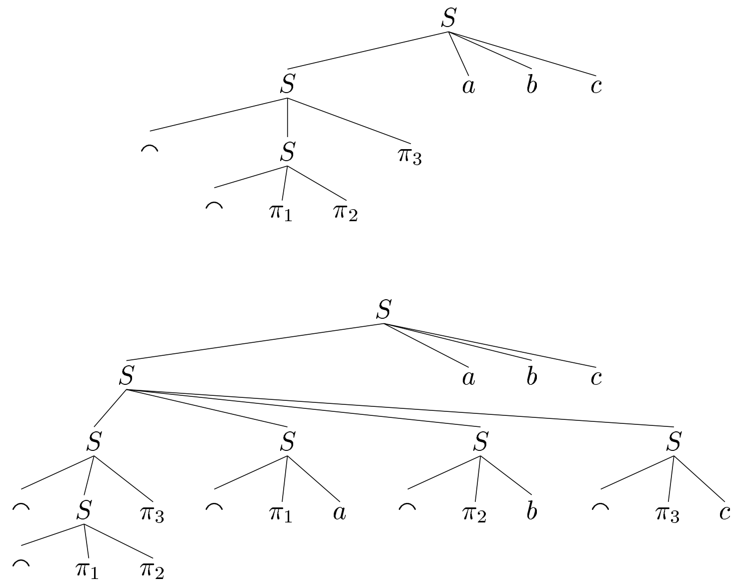


FIGURE 4. Terminal derived trees corresponding to the example in Figure 2.

regular counterpart. Before we prove a claim about this relationship in the proposition below a short historical remark appears to be apposite.

The central theorem in Maibaum (1974) to the effect that every context-free tree language is the image under the operation β of an effectively constructed regular language is wrong because he confounded the inside-out with the outside-in derivation mode. In the course of establishing a fixed-point characterization for context-free tree grammars in either generation mode Engelfriet and Schmidt (1977) point out this mistake and state as an immediate consequence of the fixed-point analysis of IO context-free tree grammars within the space of the power-set tree substitution algebra that each IO context-free tree language $\mathcal{L}$ is the image of a regular tree language $D(\mathcal{L})$ under the unique homomorphism from $\mathfrak{T}(D(\Sigma))$ into $\mathfrak{D}\mathfrak{T}(\Sigma, X)$ (see their Cor.4.12). This immediate consequence is but a restatement of the classical Mezei-Wright result that the equational subsets of an algebra are the homomorphic images of recognizable subsets in the initial term algebra. As formulated by Engelfriet & Schmidt, their correction of Maibaum's theorem has a distinctive declarative content whereas the original claim had a clear operational meaning. It was based on the contention that the individual derivation steps of a context-free tree grammar and its derived counterpart correspond to each other. This is the point of the next lemma.

Lemma *Suppose $G = \langle \Sigma, \mathcal{F}, S, P \rangle$ is a context-free tree grammar and $\mathcal{L}(G)$ its generated tree language. Then there is a derived regular tree grammar $G_D = \langle D(\Sigma), D(\mathcal{F}), D(S), D(P) \rangle$ such that $\mathcal{L}(G)$ is the image of $\mathcal{L}(G_D)$ under the unique homomorphism from the algebra of tree terms over $D(\Sigma \cup \mathcal{F})$ into the tree substitution algebra over the same alphabet. In particular, t' is derived in G from t in k steps, i.e. $t \Rightarrow t'$ via the productions $p_1, \ldots, p_k$ in P if and only if there are productions $p'_1, \ldots, p'_k$ in $D(P)$ such that $\mathsf{LIFT}(t')$ is derived in G_D from $\mathsf{LIFT}(t)$ via the corresponding productions.*

Proof. The proof is based on the closure of inside-out tree languages under tree homomorphisms. The idea of using the LIFT operation for the simulation of derivation steps on the derived level can also be found in Engelfriet and Schmidt (1978). Let h_n be a family of mappings $h_n : \Sigma_n \longrightarrow T(\Omega, X)$ where Σ and Ω are two ranked alphabets. Such a family induces a tree homomorphism $\hat{h} : T(\Sigma) \longrightarrow T(\Omega)$ according to the recursive stipulations:

$$\begin{array}{rcll} \hat{h}(\sigma) & = & h_0(\sigma) & \text{for } \sigma \in \Sigma_0 \\ \hat{h}(\sigma(t_1, \ldots, t_n)) & = & h_n(\sigma)[\hat{h}(t_1), \ldots, \hat{h}(t_n)] & \text{for } \sigma \in \Sigma_n \end{array}$$

A production p in P can be viewed as determining such a tree homomorphism $\hat{p} : T(\Sigma \cup \mathcal{F}) \longrightarrow T(\Sigma \cup \mathcal{F})$ by considering the family of mappings $p_n : \Sigma_n \cup \mathcal{F}_n \longrightarrow T(\Sigma \cup \mathcal{F}, X_n)$ where $p_n(F) = t$ for $t \in T(\Sigma \cup \mathcal{F}, X_n)$ and $p_n(f) = f(x_1, \ldots, x_n)$ for $f \neq F$ in $\Sigma \cup \mathcal{F}$. By requiring that $\hat{p}(x_i) = x_i$ the mapping $\hat{p}$ can be regarded as a $D(\Sigma \cup \mathcal{F}')$-homomorphism from the tree substitution algebra $\mathfrak{D}\mathfrak{T}(\Sigma \cup \mathcal{F}, X)$ into itself, where we have set $\mathcal{F}' := \mathcal{F} \setminus \{F\}$. By applying the LIFT-operation to the tree homomorphism $\hat{p}$ we obtain its simulation $\hat{p}_D : T(D(\Sigma \cup \mathcal{F})) \longrightarrow T(D(\Sigma \cup \mathcal{F}))$ on the derived level:

$$\begin{array}{rcll} p_{D_0}(F) & = & \mathsf{LIFT}_n(p_n(F)) & \\ p_{D_0}(f) & = & f & \text{for } f \neq F \text{ in } \Sigma \cup \mathcal{F} \\ p_{D_0}(\pi_i^n) & = & \pi_i^n & \\ p_{D_{n+1}}(S_{n,m}) & = & S_{n,m}(x_1, \ldots, x_{n+1}) & \end{array}$$

Observe, that we have treated $D(\Sigma \cup \mathcal{F})$ as a ranked alphabet. If we can show that the diagram below commutes the claim in the lemma follows by induction:

$$\begin{array}{ccc} \mathfrak{T}(D(\Sigma \cup \mathcal{F})) & \xrightarrow{\hat{p}_D} & \mathfrak{T}(D(\Sigma \cup \mathcal{F})) \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle\beta} \\ \mathfrak{D}\mathfrak{T}(\Sigma \cup \mathcal{F}, X) & \xrightarrow[\hat{p}]{} & \mathfrak{D}\mathfrak{T}(\Sigma \cup \mathcal{F}, X) \end{array}$$

The commutability is shown by the succeeding series of equations in which the decisive step is justified by the identity of $\beta \circ \mathsf{LIFT}$ on the tree substitution algebra. Let f be in $(\Sigma \cup \mathcal{F})_n$:

$$\begin{array}{rcl} \beta(\hat{p}_D(f)) & = & \beta(\mathsf{LIFT}_n(p_n(f))) \\ & = & p_n(f) \\ & = & \hat{p}(f(x_1, \ldots, x_n)) \\ & = & \hat{p}(\beta(f)). \end{array}$$

□

The preceding result provides an operational handle on the correspondence between the derivation sequences within the tree substitution algebra and the derived term algebra. As characterized, the correspondence is not of much help in finding a solution to the problem of giving a logical description of the exact computing power needed to analyze natural language phenomena. The formal definition of the β transformation, which mediates the correspondence, is of an appealing perspicuity, but the many structural properties that are exemplified in the range of this mapping make it difficult to estimate the computing resources necessary to establish the correspondence relation between input trees and their values in the semantic domain of the tree substitution algebra. We know from the classical result for regular tree languages that monadic second-order logic is too weak to serve as a logical means to define the range of the β mapping when it is applied to the space of regular tree languages. What does not seem to be excluded is the possibility of solving our logical characterization problem by defining the range of context-free tree languages *within* the domain of the regular languages. In this way, we would rest on firm ground and would take a glimpse into unknown territory by using the same logical instruments that helped us to survey our "recognizable" homeland.

4. Logical Characterization

Extending the characterization of grammatical properties by monadic second-order logic has been our main motivation. For tree languages the central result is the following: a tree language is definable in monadic second-order logic if and only if it is a regular language. As is well known and as we will indicate below, a similar characterization holds for regular and context-free string languages. The examples and analyses of English syntactic phenomena that are presented in Rogers (1994) make it abundantly clear that monadic second-order logic can be used as a flexible and powerful specification language for a wide range of theoretical constructs that form the core of one of the leading linguistic models. Therefore, the logical definability of structural properties that are coextensive with the empirically testified structural variation of natural languages, would be a useful and entirely grammar independent characterization of the notion of a possible human language.

It follows from the cross-serial dependencies in Swiss German and related phenomena in other languages that monadic second-order logic, at least in the form that subtends the characterization results just mentioned, is not expressive enough to allow for a logical solution of the main problem of Universal Grammar: determining the range in cross-language variation. In many minds, this expressive weakness alone disqualifies monadic second-order logic from consideration in metatheoretic studies on the logical foundations of linguistics. Employing the results of the preceding section, we will sketch a way out of this quandary, inspired by a familiar logical technique of talking in one structure about another. To what extent one can, based on this technique, simulate transformations of trees is not yet fully understood. We shall sketch some further lines of research in the concluding remarks.

The major aim of descriptive complexity theory consists in classifying properties and problems according to the logical complexity of the formulas in which they are expressible. One of the first results in this area of research is the definability of the classes of regular string and tree languages by means of monadic second-order logic. The use of this logic is of particular interest since it is powerful enough to express structural properties of practical relevance in the empirical sciences and it remains, in spite of its practical importance, efficiently solvable. As a preparation for our logical description of the β transformation we shall recall some concepts needed for expressing the logical characterization of the class of regular languages.

For the purpose of logical definability we regard strings over a finite alphabet as model-theoretic structures of a certain kind.

DEFINITION 14. *Let Σ be an alphabet and let $\tau(\Sigma)$ be the vocabulary $\{<\} \cup \{P_a \mid a \in \Sigma\}$, where $<$ is binary and the P_a are monadic. A* ***word model*** *for $u \in \Sigma^*$ is a structure of the form*

$$\mathcal{M} = (B, <, P_a)$$

where $|B| = length(u)$, $<$ is an ordering of B and the P_a correspond to positions in u carrying the label a:

$$P_a = \{b \in B \mid label(b) = a\}.$$

The corresponding monadic second-order logic $\mathsf{MSOL}(\Sigma)$ has *node variables* $x, y, z, \ldots$ and *set variables* $X, Y, Z, \ldots$ ranging over subsets of the domain of discourse B. There are four kinds of atomic formulas $x = y$, $x < y$, $P_a(x)$ and $x \in X$. The formulas of the language are constructed from the atomic formulas in the expected way by combining them by means of the usual propositional connectives and the existential and universal quantifiers. Observe that not only individual node variables, but also node-set variables may be bound by quantifiers. For a closed formula A in this language and a word model $\mathcal{M}$, we write $\mathcal{M} \models A$ if A is true in M. The set of models that make a closed formula A true is denoted by $Mod(A)$. A theorem by Elgot talks about the close relationship between model classes definable by closed formulas of $\mathsf{MSOL}(\Sigma)$ and regular string languages.

THEOREM 1 (Elgot). *$\mathcal{L}$ is regular over the alphabet Σ iff $\mathcal{L}$ is definable in monadic second-order logic over the vocabulary $\tau(\Sigma)$. More succinctly:*

$$\mathcal{L} \in \mathcal{L}_3 \textit{ iff } \exists A \in \mathsf{MSOL}(\Sigma) \textit{ such that } \mathcal{L} = Mod(A) = \{u \in \Sigma^* \mid u \models A\}.$$

For the statement of this classical result for tree languages we need an appropriate notion of model-theoretic structure. If Σ is a finite ranked alphabet, a tree in $T(\Sigma)$ is coded by a model of the following form, which is called a *labeled tree model.*

DEFINITION 15. *Let Σ be a finite ranked alphabet and let $\tau(\Sigma)$ be the vocabulary $\{<_i\}_{i\leq n} \cup \{P_a \mid a \in \Sigma\}$, where each $<_i$ is binary and each P_a is a monadic predicate. A **labeled tree model** for $t \in T_\Sigma$ is a structure of the form*

$$M = (B, <_i, P_a)_{i\leq n, a\in\Sigma}$$

where B is in one-one correspondence with $dom(t)$, the tree domain of t, $<_i$ is the i-th successor function, $n =$ the maximal m such that $\Sigma_m \neq \emptyset$ and the P_a correspond to positions with label a:

$$P_a = \{b \in B \mid label(b) = a\}.$$

The notion of a labeled tree is built upon a system of nodes or tree addresses which are strings in $\mathbb{N}^*$ and which satisfy two closure properties. More precisely, the *domain* of a finite ordered tree with at most n outgoing branches is a finite subset $D \subseteq \mathbb{N}_n^*$, where $\mathbb{N}_n = \{0, \ldots, n-1\}$, such that the following conditions hold:

(i) If $uv \in D$ $(u, v \in \mathbb{N}_n^*)$ then $u \in D$.
(ii) If $ui \in D$ and $j < i$ then $uj \in D$ $(u \in \mathbb{N}_n^*,\ i \in \mathbb{N}_n)$.

The appropriate *monadic second-order logic of n successors* $\mathsf{MSOL}_n(\Sigma)$ has *node variables* $x, y, z, \ldots$ and *set variables* $X, Y, Z, \ldots$, ranging over subsets of the tree domain. There are again four kinds of atomic formulas $x = y$, $x <_i y$, $P_a(x)$ and $x \in X$, where we have presented the successor function in relational form. Arbitrary formulas are generated from atomic formulas by propositional connectives and the quantifiers binding both kinds of variables. If A is an $\mathsf{MSOL}_n(\Sigma)$ sentence, i.e. a formula without free variables, and $\mathcal{M}$ a tree model, we write $\mathcal{M} \models A$ to indicate that A is true in $\mathcal{M}$. If $Mod(A) = \{\mathcal{M} \mid \mathcal{M} \models A\}$, we call this set of trees *definable* in $\mathsf{MSOL}_n(\Sigma)$. The next theorem states the extension of Elgot's result to the yield or frontier of finite ranked ordered trees.

THEOREM 2 (Doner, Thatcher–Wright). *Let Σ be a ranked alphabet.*

a) *$\mathcal{L} \subseteq \Sigma^*$ is **context-free** iff $\mathcal{L}$ is the yield of a set of tree models definable in monadic second-order logic of n successors over the vocabulary $\tau(\Sigma)$.*

$\mathcal{L} \in \mathcal{L}_2$ iff $\exists A \in \mathsf{MSOL}_n(\Sigma)$ such that $\mathcal{L} = yield(Mod(A))$

where $Mod(A) = \{t \in T_\Sigma \mid t \models A\}$.

b) *A set of trees is generated by a regular tree grammar G iff there is a sentence A in $\mathsf{MSOL}_n(\Sigma)$ such that*

$$Mod(A) = \mathcal{L}(G).$$

Given the intimate relationship between regular tree languages and context-free string languages on the one hand and monadic second-order logic on the other, there is no hope to address the question as to how to account for context-sensitive formalisms with the definitional resources of this logic directly. An alternative approach is suggested by the characterization of the context-free tree languages as the class of languages that is the homomorphic image of the class of regular tree languages over a related alphabet. This characterization says that the context-free tree languages constitute the class of all tree families $Tr(\mathcal{L})$ where $\mathcal{L}$ is a subset of $T(D(\Sigma))$ and Tr is a function from explicit trees over $D(\Sigma)$ to trees over Σ.

The β transformation that sends regular tree languages to their context-free counterparts is a particular instance of a relation that mediates between finite trees over different alphabets. If Σ and Ω are two alphabets, then a binary relation $Tr \subseteq T(\Sigma) \times T(\Omega)$ is generally called a *tree transduction*. Tree transductions can be interpreted as transforming a tree over one alphabet into a tree over another alphabet. Of special interest are those tree transductions that can be specified in an effective way and among these, the top-down and bottom-up transductions, performed by finite automata that process a tree from the root down or from the leaves up, have been thoroughly investigated since the early 70's. It has been shown that these types of transductions do not preserve regularity, but that their range is a proper subset of the context-free tree languages when their domain is restricted to the regular tree languages. Since the range of the β transformation is the full range of context-free tree languages under the same domain restriction its action cannot be simulated by a top-down or a bottom-up tree transduction.

As the reader will recall, the transformations performed by the β function interpret the operators S and π as substitution and projection, respectively. This behaviour is of such a formal simplicity that an effective description of its structural pattern should be possible, even though a simulation by a classical (top-down or bottom-up) tree automaton is excluded according to the result just mentioned. Trees in $T(D(\Sigma))$ and trees in $T(\Sigma)$ are really different ways of looking at one and the same thing. Therefore, it would further the project of an effective description of tree classes in the substitution algebra if there was an effective syntactic map from the vocabulary $D(\Sigma)$ into the vocabulary Σ, because we know already that monadic second-order logic provides an effective medium to define the regular tree classes in the initial $D(\Sigma)$-algebra.

There is an established technique in model theory that enables one to show that in certain circumstances the choice of language is arbitrary. The central notion employed in this technique is that of a syntactic interpretation. The interesting feature of *syntactic interpretation* is that it allows one to talk in one structure about another, as we will now attempt to show. Suppose we have two relational signatures Σ and Ω with no constant or function symbols and, in addition to a *domain formula* $A_\Omega(x)$, for each n-ary $R \in \Omega$ a *defining formula* $A_R(x_1, \ldots, x_n)$, where these formulas are built from the symbols in Σ. Given a Σ-structure $\mathcal{M}$ in which the domain formula $A_\Omega(x)$ defines the universe of discourse of an Ω-structure $\mathcal{M}'$, it can then be shown that for every Ω-formula B one can find an Σ-formula B' such that

$$\mathcal{M} \models B'\,[g] \text{ iff } \mathcal{M}' \models B\,[g]$$

where g is a variable assignment in $\mathcal{M}'$.

Note, that from a different perspective the domain formula $A_\Omega(x)$ can be viewed as establishing a binary relation between Σ- and Ω-structures in the following way. We let A_Ω single out those Σ-structures where its satisfaction set is nonempty:

$$\mathcal{M}' = \{a \,|\, \mathcal{M} \models A_\Omega[a]\} \neq \emptyset.$$

On $\mathcal{M}'$ an Ω-structure is easily specified by the following stipulation for each $R \in \Omega$:

$$\begin{aligned} R_{\mathcal{M}'} := \{(a_1, \ldots, a_n) \,|\, \mathcal{M} \models A_R[a_1, \ldots, a_n]\ \&\\ \mathcal{M} \models A_\Omega[a_i] \text{ for } i = 1, \ldots, n\} \end{aligned}$$

(In case Ω contains any functional relations appropriate closure conditions have to be required in addition to the non-emptiness of $\mathcal{M}'$.) As defined, A_Ω is nothing but a binary relation $A_\Omega \subseteq \mathbf{M}_\Sigma \times \mathbf{M}_\Omega$ where $\mathbf{M}_\Sigma$ denotes the class of Σ-structures and $\mathbf{M}_\Omega$ the class of Ω-structures: Two structures $\mathcal{M}$ and $\mathcal{M}'$ stand in the relation A_Ω if $\mathcal{M}$ is a Σ-structure in which we can talk about an internal Ω-structure $\mathcal{M}'$ that is carved out by the satisfaction of A_Ω.

The above analysis involves the obvious suggestion to provide an account of tree transductions within the model-theoretic framework of syntactic interpretation. Once labeled trees are looked at as specific model-theoretic structures a tree transduction Tr becomes a relation between such structures in perfect analogy to the case of relational structures connected by a set of defining and domain formulas.

The first to see the potential of applying the method of syntactic interpretation to the analysis of relations between graph-like structures was B. Courcelle. He introduces for this purpose the notion of a monadic second-order definable graph transduction. The switch from first-order logic to monadic second-order logic is caused by the same reasons that were given above for the adoption of monadic second-order logic in the case of labeled ordered trees. Apart from this recourse to higher-order logic Courcelle's definition is patterned upon the classical format provided by the method of syntactic interpretation. Let us record here its adaptation to tree structures.

DEFINITION 16. *Let Σ and Ω be tow finite ranked alphabets. A* ***monadic second-order definable tree transduction*** *is defined as follows. We fix a syntactic interpretation $I = (A, A_\Omega(x), A_{<_i}(x,y), A_{P_a}(x))_{i\leq n, a\in\Omega}$ consisting of a tuple of monadic second-order formulas over the signature Σ. These formulas are intended to define a tree model t' in $T(\Omega)$ from a tree model t in $T(\Sigma)$. The closed formula A is to define the domain of the transduction, the formula $A_\Omega(x)$ is to define the tree domain of t' and the formulas $A_{<_i}(x,y)$ and $A_{P_a}(x)$ are to define the successor relations and the distribution of node labels on the tree domain $dom(t')$. n indicates the maximum arity of a symbol in Ω.*

A tree model t' in $T(\Omega)$ with tree domain D is defined in $t \in T(\Sigma)$ by I if

a) (i) $t \models A$
   (ii) $D = \{u \mid t \models A_\Omega\,[u]\}$
   (iii) $<_{i_{t'}} = \{(u,v) \in D^2 \mid t \models A_{<_i}\,[u,v]\}$
   (iv) $P_{a_{t'}} = \{u \in D \mid t \models A_{P_a}\,[u]\}$

b) *D satisfies the conditions of a tree domain with $<_{i_{t'}}$ its successor relations.*

Indicating this last (functional) relation by $def_I(t) = t'$ we can finally say what is denoted by the monadic second-order tree transduction defined by I:

$$def_I = \{(t,t') \mid def_I(t) = t'\}.$$

As defined, a definable tree transduction is indeed a tree transduction in the sense of the abstract transformation relation $Tr \subseteq T(\Sigma)\times T(\Omega)$ between tree domains that was introduced above. What is more, definable transductions possess the sort of computational properties that we asked of an effective syntactic mapping. The syntactic interpretation is a finitary object, consisting of a tuple of formulas, and all the conditions in the definition are expressible by formulas of the monadic second-order language, which are decidable in general and even testable in linear time on given tree models. In other words, the co-domain of a definable transduction def_I may not be definable itself, i.e. there may be no monadic second-order formula A' such that $\{t \mid t \models A'\} = \{t' \mid def_I(t) = t' \text{ for some } t \text{ s.t. } t \models A\}$ where A defines the domain of def_I. The second-order theory of the co-domain of a definable transduction def_I, on the other hand, is decidable and we can call it even definable if its domain formula is admitted as a restricting parameter.

Definable tree transductions would thus provide a solution to the logical characterization problem if the type of tree transformation performed by the β mapping could be given a logical definition via a syntactic interpretation scheme. This is fortunately the case as stated in the next theorem.

THEOREM 3. *The β transformation of a regular tree language is a monadic second-order definable tree translation.*

Proof Sketch. The detailed verification of the theorem's claim is left for another occasion. Let us point out that it can be reduced to the definability of the yield function of graph grammars with neighbourhood controlled embedding and dynamic edge relabeling (edNCE grammars). Productions in this type of grammar are of the form $X \longrightarrow (D, C)$ where X is an operative or functional node label, D is a graph and C is a set of connection instructions. A derivation step licensed by such a production consists in replacing an X-labeled node by D and connecting D to the neighbourhood of X according to the specifications in C. As will be recalled, a much more specific form of local node replacement played a central role in macro-like productions of context-free tree grammars. It is straightforward to associate a derivation tree with a derivation in a edNCE grammar. A derivation is a labeled ordered tree in which the nodes are labeled with productions. If a node has the label $X \longrightarrow (D, C)$ its daughters are labeled by a sequence of production rules whose left-hand sides exhaust the operative nodes in the replacing graph D. The *yield* function executes the substitutions of the right-hand sides of the production rules in a bottom-up regime, observing along the way the connecting restrictions. The yield of the derivation tree returns as result the graph that is generated by a corresponding derivation of the edNCE grammar.

The fact that the yield function of an edNCE grammar is definable in monadic second-order logic was proved by van Oostrom and presented in a generalized form in Courcelle (1992). The trees in a regular tree grammar over a derived alphabet can be read as a special form of derivation trees in the sense of edNCE graph grammars. In these special trees, the replacements of the functional symbols have already been carried out, but the connecting instructions which are coded in the explicit substitution and projection labels have to be executed by the β transformations. Thus, the β transformation turns out to be a special case of the yield function, which means that the β transformation can be simulated by a definable transduction. A detailed proof of the claim that the case of context-free tree languages is indeed covered by the context-free neighbourhood controlled graph languages will appear elsewhere. □

5. Concluding Remarks

In the present paper, we have tried to combine the methods developed in the tradition of algebraic language theory with the techniques of model theory. While there is a perfect match between these two methodological frameworks in the case of regular or, equivalently, taking the automata-theoretic view, recognizable families of trees, this state of harmony is seriously disturbed once one considers the far richer family of context-free sets of trees. We had no choice but to take these richer structures into account, since a series of well documented non-local dependencies in natural languages fall outside the range of configurational properties that can be accommodated within the realm of regular trees. The grammar independent approach to characterizing syntactic concepts has been so rewarding, on the other hand, that it seemed worthwhile to make a serious attempt at providing the formal background to extend the tools of the logical analysis beyond the harmonious domain of strictly local dependencies.

As was emphasized several times above, it may well be possible to strengthen the language of monadic second-order logic without losing its algorithmic tractability. Whether such a strengthening would create a framework for a direct application of monadic second-order logic's definitional resources to the domain of context-free tree languages remains a challenging open problem.

Deterred by the apparent obstacles to devising a manageable logical specification language for non-local dependencies we opted for the timid alternative, which consists in forming a picture of the outer world in terms of the conceptual network that fits so smoothly the regular homeland. This description of our approach refers both to the algebraic point of view according to which we evaluate (uniquely) the regular term expressions in the semantic domain of context-free tree languages and to the logical point of view according to which we consider the recognizable forest of trees as relational structures in the model-theoretic sense and employ their language to talk about the foreign country where non-local dependencies roam.

It is notoriously difficult to tell to what an extent the use of our own vernacular gives a distorted picture of a foreign country. The complexity of the clauses that enter into an explicit definition of the evaluation transduction would give an idea of how faithful our language reflects the situation in an area where it has not been used as a proven means of communication. We had to postpone the task of spelling out the translation scheme involved in such an evaluating transduction but the reader will be able to measure the distance between a syntactic configuration under its intended representation and its "regular" counterpart after a perusal of H. Kolb's contribution to the present volume.

There is a last disclaimer that needs to be made. The term definable transduction imports the misleading notion of an operational transformation of structures. This misinterpretation would bestow a power upon the device of syntactic interpretation that it definitely does not have. The issue of whether certain linguistic phenomena are more easily described in a derivational rather than a representational model should not be confused with the issue of whether it is possible to give a direct logical definition of context-free tree structures or only an indirect definition via a definable transduction. The fact that we have opted for an indirect solution to our problem testifies only to our lack of ingenuity and not to our allegiance to the derivationists' cause.

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