Algebraic Description of Derivational Minimalism

Jens Michaelis
Uwe Mönnich
Frank Morawietz

Seminar für Sprachwissenschaft
Universität Tübingen

{michael,um,frank}@sfs.nphil.uni-tuebingen.de

http://tcl.sfs.nphil.uni-tuebingen.de/

Iowa City, 22.05.2000

Outline

- Motivation and Theoretical Background
- Derivational Minimalism
- From Minimalism to Multiple Context-Free Grammars
- From MCFGs to Regular Tree Grammars
- Coding of the Intended Structures

Linguistic Background

J. Rogers (1994,98) gives a formalization within MSO tree logic of a main paradigm in Theoretical Linguistics:

The Principle and Parameters Approach

in the tradition of N. Chomsky

+ Natural Formalization of (almost) all Principles
+ Modular
+ Decidable
+ Descriptive Complexity Result
− Undecidability of *Free Indexation* (Grid)
− Non Context-Free Phenomena in Natural Language

Rogers (1994,1998): $\mathbf{L^2_{K,P}}$

- K a set of individual constants
- P a set of predicate constants
- immediate dominance: $\lhd$
- reflexive dominance: $\lhd^*$
- proper dominance: $\lhd^+$
- proper precedence: $\prec$
- usual connectives, variables, equality, set membership and quantifiers:
  - a set of *monadic* second-order variables (X,Y,...);
  - a set of individual variables (x,y,...);
  - boolean connectives: $\neg$, $\wedge$, ...;
  - Quantifiers over individuals and sets: $\exists$, $\forall$;
  - equality and membership: $\approx$, $\in$.

Linguistic Motivation

- Non-Contextfree Phenomena:

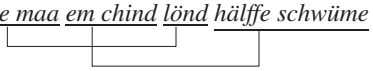
  (... weil) der Karl die Maria dem Peter den Hans schwimmen lehren helfen liess

  German: Palindrome language—CF

  (... omdat) Karel Marie Piet Jan liet helpen leren zwemmen

  Dutch: $a^n b^n$—CF

  (... wil) mer de maa em chind lönd hälffe schwüme

  Swiss-German: $a^n b^m c^n d^m$—Non-CF

- Derivational Minimalism is Non-Contextfree

Formal Background

Weak Equivalence of Mildly Context-Sensitive Formalisms:

STR(HR): languages generated by string generating hyperedge replacement grammars

OUT(DTWT): output languages of deterministic tree-walking tree-to-string transducers

yDT$_{\mathbf{fc}}$(REGT): yields of images of the regular tree languages under deterministic finite-copying top-down tree transductions

MCFL: languages generated by multiple context-free grammars

MCTAL: languages generated by multi-component tree adjoining grammars

LCFRL: languages generated by linear context-free rewriting systems

LUSCL: languages generated by local unordered scattered context grammars

General Structure of the Approach

Translating Minimalist Grammars to MCFGs

$\Rightarrow$ Regular Characterization of the MCFG-Languages

$\Rightarrow$ Retrieval of the Intended Structures via Regular Relations

Minimalist Grammars (Stabler, 1997)

$$\mathcal{MG} = \langle NonSyn, Syn, Lex, F \rangle$$

$$\begin{array}{lcl} NonSyn & = & Phon \cup Sem \\ Syn & = & \mathsf{base} \cup \mathsf{select} \cup \mathsf{licensee} \cup \mathsf{licensor} \\ Lex & = & \text{a finite set of trees} \\ F & = & \{\text{MERGE}, \text{MOVE}\} \end{array}$$

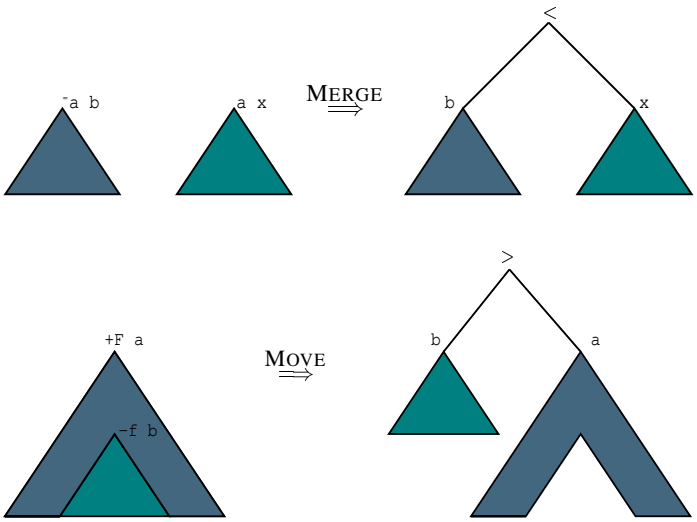
Phon	are the phonological features
Sem	are the interpreted/semantical features

$$\begin{array}{lcl} \mathsf{base} & = & \{\texttt{c}, \texttt{t}, \texttt{v}, \texttt{d}, \texttt{n}, \ldots\} \\ \mathsf{select} & = & \{\texttt{=x}, \texttt{=X}, \texttt{X=} \mid \texttt{x} \in \mathsf{base}\} \\ \mathsf{licensee} & = & \{\texttt{-case}, \texttt{-wh}, \ldots\} \\ \mathsf{licensor} & = & \{\texttt{+case}, \texttt{+CASE}, \texttt{+wh}, \texttt{+WH}, \ldots\} \end{array}$$

$$Label : Leaves \longrightarrow \mathsf{select}^*(\mathsf{licensor})\mathsf{select}^*(\mathsf{base})\mathsf{licensee}^* P^* I^*$$

$$\mathcal{L}_{\mathcal{MG}} = Closure(Lex, F)$$

MERGE/MOVE



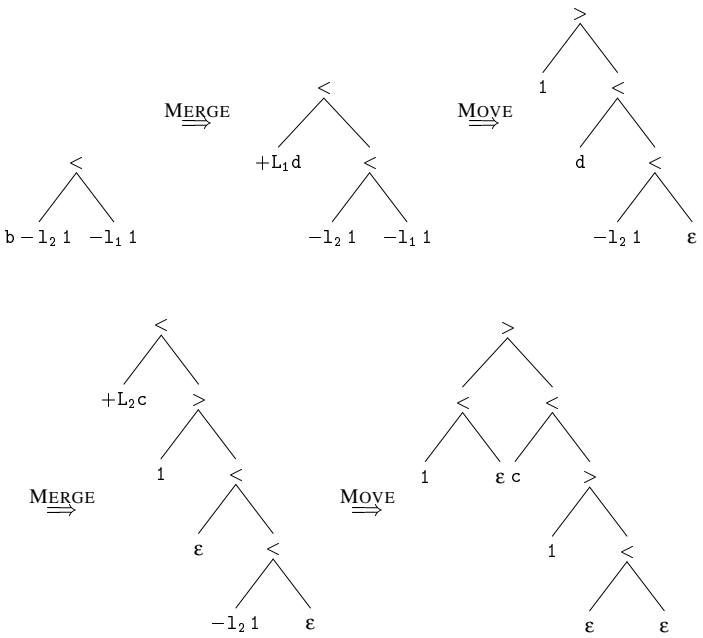
An Example MG: $G_{ww} = \langle NonSyn, Syn, Lex, F \rangle$

$$Sem = \emptyset$$
$$Phon = \{1, 2\}$$
$$base = \{\mathtt{c}, \mathtt{a}_1, \mathtt{a}_2, \mathtt{b}, \mathtt{c}_1, \mathtt{c}_2, \mathtt{d}\}$$
$$select = \{\mathtt{=a}_1, \mathtt{=a}_2, \mathtt{=b}, \mathtt{=c}_1, \mathtt{=c}_2, \mathtt{=d}\}$$
$$Licensors = \{\mathtt{+L}_1, \mathtt{+L}_2\}$$
$$Licensees = \{\mathtt{-l}_1, \mathtt{-l}_2\}$$

$$Lex : i \in \{1, 2\}$$

$$\alpha_i = \mathtt{a}_i\mathtt{-l}_1 i \quad \gamma_i = \mathtt{=b+L}_1\mathtt{c}_i\mathtt{-l}_1 i \quad \zeta_1 = \mathtt{=b+L}_1\mathtt{d}$$
$$\beta_i = \mathtt{=a}_i\mathtt{b-l}_2 i \quad \delta_i = \mathtt{=c}_i\mathtt{+L}_2\mathtt{b-l}_2 i \quad \zeta_2 = \mathtt{=d+L}_2\mathtt{c}$$

An Example Derivation



Derivational Minimalism is mildly context-sensitive

Theorem (Michaelis). Each Minimalist Grammar (MG) in the sense of Stabler (1997) can be translated into a weakly equivalent multiple context-free grammar (MCFG) and is therefore mildly context-sensitive.

Idea of the Proof:

- define a finite partition
- each euqivalence class is a set of trees such that the features triggering movement appear in identical structural positions
- each nonterminal in the MCFG represents such an equivalence class

MCFG

$$\mathcal{G} = \langle V_N, V_T, V_F, P, S \rangle$$

V_N	is a ranked set of nonterminals
V_T	is a ranked set of terminals
V_F	are linear basic morphisms
S	$\in V_N$ is the start symbol
P	are the productions

$$A \longrightarrow f(A_0, \ldots, A_{n-1})$$
$$A, A_0, \ldots, A_{n-1} \in V_N$$
$$f \in V_F : (T^*)^k \longrightarrow (T^*)^l$$
$$k = \Sigma_{i=0}^{n-1} k_i \qquad (k_i \text{ the rank of } A_i)$$
$$l \text{ the rank of } A$$

An Example Grammar $\mathcal{G}_{ww}$: (Non)terminals

$$V_T = Phon = \{1,2\}$$

$$U = \langle \texttt{b-l}_2, \texttt{-l}_1, - \rangle \rightsquigarrow \{\omega_i \mid i \in \{1,2\}\}$$
$$V_i = \langle \texttt{+L}_1\texttt{c}_i\texttt{-l}_1, \texttt{-l}_1, \texttt{-l}_2 \rangle \rightsquigarrow \{\varphi_{iu} \mid u \in \{1,2\}^+, i \in \{1,2\}\}$$
$$U_i = \langle \texttt{c}_i\texttt{-l}_1, -, \texttt{-l}_2 \rangle \rightsquigarrow \{\chi_{iu} \mid u \in \{1,2\}^+, i \in \{1,2\}\}$$

$$V = \langle \texttt{+L}_2\texttt{b-l}_2, \texttt{-l}_1, \texttt{-l}_2 \rangle \rightsquigarrow \{\psi_{iu} \mid u \in \{1,2\}^+, i \in \{1,2\}\}$$
$$U = \langle \texttt{b-l}_2, \texttt{-l}_1, - \rangle \rightsquigarrow \{\omega_{iu} \mid u \in \{1,2\}^+, i \in \{1,2\}\}$$
$$W_1 = \langle \texttt{+L}_1\texttt{d}, \texttt{-l}_1, \texttt{-l}_2 \rangle \rightsquigarrow \{\eta_u \mid u \in \{1,2\}^+\}$$
$$X_1 = \langle \texttt{d}, -, \texttt{-l}_2 \rangle \rightsquigarrow \{\vartheta_u \mid u \in \{1,2\}^+\}$$
$$W_2 = \langle \texttt{+L}_2\texttt{c}, -, \texttt{-l}_2 \rangle \rightsquigarrow \{\kappa_u \mid u \in \{1,2\}^+\}$$
$$C = \langle \texttt{c}, -, - \rangle \rightsquigarrow \{\xi_u \mid u \in \{1,2\}^+\}$$

An Example Grammar $\mathcal{G}_{ww}$: Rules

$$i \in \{1,2\}.$$

$$U \to merge_{\langle \beta_i, \alpha_i \rangle}()$$

$$V_i \to merge_{\langle \gamma_i, \omega_ \rangle}(U) \qquad W_1 \to merge_{\langle \zeta_1, \omega_ \rangle}(U)$$
$$U_i \to move_{\langle \varphi_ \rangle}(V_i) \qquad X_1 \to move_{\langle \eta_ \rangle}(W_1)$$
$$V \to merge_{\langle \delta_i, \chi_ \rangle}(U_i) \qquad W_2 \to merge_{\langle \zeta_2, \vartheta_ \rangle}(X_1)$$
$$U \to move_{\langle \psi_ \rangle}(V) \qquad C \to move_{\langle \kappa_ \rangle}(W_2)$$

$$S \to \pi_1^3(C)$$

An Example Grammar $\mathcal{G}_{ww}$: Basic Functions

$$merge_{\langle \beta_i, \alpha_i \rangle} : (\{1,2\}^*)^0 \to (\{1,2\}^*)^3 \text{ with } \langle\rangle \mapsto \langle i, i, \varepsilon \rangle$$
$$merge_{\langle \gamma_i, \omega_ \rangle} : (\{1,2\}^*)^3 \to (\{1,2\}^*)^3 \text{ with } \langle x_1, x_2, x_3 \rangle \mapsto \langle i, x_2, x_1 \rangle$$
$$move_{\varphi_} : (\{1,2\}^*)^3 \to (\{1,2\}^*)^3 \text{ with } \langle x_1, x_2, x_3 \rangle \mapsto \langle x_2 x_1, \varepsilon, x_3 \rangle$$
$$merge_{\langle \delta_i, \chi_ \rangle} : (\{1,2\}^*)^3 \to (\{1,2\}^*)^3 \text{ with } \langle x_1, x_2, x_3 \rangle \mapsto \langle i, x_1, x_3 \rangle$$
$$move_{\psi_} : (\{1,2\}^*)^3 \to (\{1,2\}^*)^3 \text{ with } \langle x_1, x_2, x_3 \rangle \mapsto \langle x_3 x_1, x_2, \varepsilon \rangle$$

$$merge_{\langle \zeta_1, \omega_ \rangle} : (\{1,2\}^*)^3 \to (\{1,2\}^*)^3 \text{ with } \langle x_1, x_2, x_3 \rangle \mapsto \langle \varepsilon, x_2, x_1 \rangle$$
$$move_{\eta_} : (\{1,2\}^*)^3 \to (\{1,2\}^*)^3 \text{ with } \langle x_1, x_2, x_3 \rangle \mapsto \langle x_2 x_1, \varepsilon, x_3 \rangle$$
$$merge_{\langle \zeta_2, \vartheta_ \rangle} : (\{1,2\}^*)^3 \to (\{1,2\}^*)^3 \text{ with } \langle x_1, x_2, x_3 \rangle \mapsto \langle x_1, \varepsilon, x_3 \rangle$$
$$move_{\kappa_} : (\{1,2\}^*)^3 \to (\{1,2\}^*)^3 \text{ with } \langle x_1, x_2, x_3 \rangle \mapsto \langle x_3 x_1, x_2, \varepsilon \rangle$$
$$\pi_1^3 : (\{1,2\}^*)^3 \to \{1,2\}^* \text{ with } \langle x_1, x_2, x_3 \rangle \mapsto x_1$$

An example derivation

$$
\begin{aligned}
& S \\
\rightarrow\ & \pi_1^3(C) \\
\rightarrow\ & \pi_1^3(move_{\langle \kappa_ \rangle}(W_2)) \\
\rightarrow\ & \pi_1^3(move_{\langle \kappa_ \rangle}(merge_{\langle \zeta_2, \vartheta_ \rangle}(X_1))) \\
\rightarrow\ & \pi_1^3(move_{\langle \kappa_ \rangle}(merge_{\langle \zeta_2, \vartheta_ \rangle}(move_{\langle \eta_ \rangle}(W_1)))) \\
\rightarrow\ & \pi_1^3(move_{\langle \kappa_ \rangle}(merge_{\langle \zeta_2, \vartheta_ \rangle}(move_{\langle \eta_ \rangle}(merge_{\langle \zeta_1, \omega_ \rangle}(U))))) \\
\rightarrow\ & \pi_1^3(move_{\langle \kappa_ \rangle}(merge_{\langle \zeta_2, \vartheta_ \rangle}(move_{\langle \eta_ \rangle}(merge_{\langle \zeta_1, \omega_ \rangle}(merge_{\langle \beta_i, \alpha_i \rangle}()))))) \\
\rightarrow\ & \pi_1^3(move_{\langle \kappa_ \rangle}(merge_{\langle \zeta_2, \vartheta_ \rangle}(move_{\langle \eta_ \rangle}(merge_{\langle \zeta_1, \omega_ \rangle}(\langle 1, 1, \varepsilon \rangle))))) \\
\rightarrow\ & \pi_1^3(move_{\langle \kappa_ \rangle}(merge_{\langle \zeta_2, \vartheta_ \rangle}(move_{\langle \eta_ \rangle}(\langle \varepsilon, 1, 1 \rangle)))) \\
\rightarrow\ & \pi_1^3(move_{\langle \kappa_ \rangle}(merge_{\langle \zeta_2, \vartheta_ \rangle}(\langle 1\varepsilon, \varepsilon, 1 \rangle))) \\
\rightarrow\ & \pi_1^3(move_{\langle \kappa_ \rangle}(\langle 1\varepsilon, \varepsilon, 1 \rangle)) \\
\rightarrow\ & \pi_1^3(\langle 11\varepsilon, \varepsilon, \varepsilon \rangle) \\
\rightarrow\ & 11\varepsilon
\end{aligned}
$$

Basic Definitions I: many-sorted signature

Let S be a set of sorts. A *many-sorted signature* Σ is an indexed family

$$\langle \Sigma_{w,s} \mid w \in S^*, s \in S \rangle$$

of disjoint sets. A symbol in $\Sigma_{w,s}$ is called an:

operator of type $\langle w, s \rangle$,

arity w,

sort s and

rank $|w|$.

In the case of a single-sorted signature we write $\Sigma_{s^n,s}$ as Σ_n.

Basic Definitions II: $T(\Sigma)$

- If $\sigma \in \Sigma_{\varepsilon,s}$ for some $s \in \mathcal{S}$ then σ is a (trivial) tree of sort s.
- If, for some $s \in \mathcal{S}$ and $w = s_1 \cdots s_n$ with $s_i \in \mathcal{S}$, $\sigma \in \Sigma_{w,s}$ and $t_1, \ldots, t_n \in T(\Sigma)$ with t_i of sort s_i then $\sigma(t_1, \ldots, t_n)$ is a tree of sort s.

Basic Definitions III: Σ-algebra

A Σ-*algebra* $\mathbb{A}$ consists of an S-indexed family of sets $A = \langle A^s \rangle_{s \in S}$ and for each operator $\sigma \in \Sigma_{w,s}$ a function

$$\sigma_{\mathbb{A}} : A^w \rightarrow A^s$$
$$A^w = A^{w_1} \times \cdots \times A^{w_n}$$
$$w = w_1 \cdots w_n.$$

The set $T(\Sigma)$ can be made into a Σ-algebra $\mathbb{T}$ by specifying the operations as follows.

For every $\sigma \in \Sigma_{w,s}$ and every $t_1, \ldots, t_n \in T(\Sigma)$ we identify

$$\sigma_{\mathbb{T}}(t_1, \ldots, t_n) \text{ with } \sigma(t_1, \ldots, t_n).$$

Lawvere Algebra I

Given a set of sorts S, a Lawvere theory, as an algebra, is an $S^* \times S^*$-sorted algebra $\mathbb{A}$, whose carriers $\langle A^{(u,v)} | u, v \in S^* \rangle$ consist of the morphisms of the theory and whose operations are of the following types ($n \in \mathbb{N}$, $u = u_1 \cdots u_n$ with $u_i \in \mathcal{S}$ for $1 \leq i \leq n$ and $v, w \in \mathcal{S}^*$):

projection: $\pi_i^u \in A^{\langle u, u_i \rangle}$

composition: $c_{(u,v,w)} \in A^{\langle u,v \rangle} \times A^{\langle v,w \rangle} \to A^{\langle u,w \rangle}$

target tupling: $(\quad)_{(v,u)} \in A^{\langle v,u_1 \rangle} \times \cdots \times A^{\langle v,u_n \rangle} \to A^{\langle v,u \rangle}$

The projections and the operations of target tupling are required to satisfy the obvious identities for products and the composition operations must satisfy associativity.

Basic Definitions IV: $T(\Sigma, X)$

Let Σ be a single-sorted signature and $X = \{x_1, x_2, x_3, \ldots\}$ a countable set of variables. For $k \in \mathbb{N}$ define $X_k \subseteq X$ as $\{x_1, \ldots, x_k\}$.

- $T(\Sigma, X_k)$ (over Σ) is the set of trees $T(\Sigma')$ over the single-sorted signature $\Sigma' = \langle \Sigma'_n \,|\, n \in \mathbb{N} \rangle$, where $\Sigma'_0 = \Sigma_0 \cup X_k$ and $\Sigma'_n = \Sigma_n$ for $n > 0$.
- $T(\Sigma, X_k) \subseteq T(\Sigma, X_l)$ for $k \leq l$.
- $T(\Sigma, X) = \bigcup_{k \in \mathbb{N}} T(\Sigma, X_k)$.

Lawvere Algebra II

With S being a singleton, the power set $\wp(T(\Sigma, X_k))$ of k-ary trees constitutes the central example of interest for formal language theory. The carriers

$$\langle \wp(T(k,m)) \,|\, k, m \in w \rangle$$

consist of the power sets of the sets $T(k,m)$ where each $T(k,m)$ is the set of all m-tuples of k-ary trees $\{(t_1, \ldots, t_m)\}$.

Regular Tree Grammars

$$\Gamma = \langle \Sigma, \mathsf{F}_0, S, \mathsf{P} \rangle \text{ where}$$

$\Sigma = \langle \Sigma_{w,s} \,|\, w \in \mathcal{S}^*, s \in \mathcal{S} \rangle$

$\mathsf{F}_0 = \langle \mathsf{F}_{\varepsilon,s} \,|\, s \in \mathcal{S} \rangle$

$S \in \mathsf{F}_0$ is the starting symbol

P is a set of productions

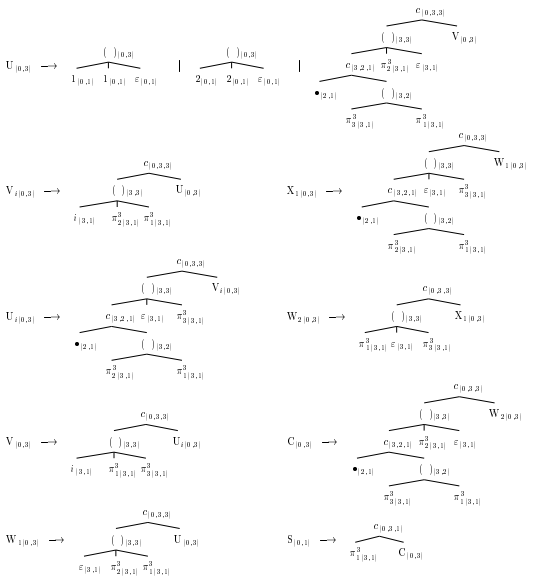
The $p \in \mathsf{P}$ have the form

$$F \longrightarrow t$$

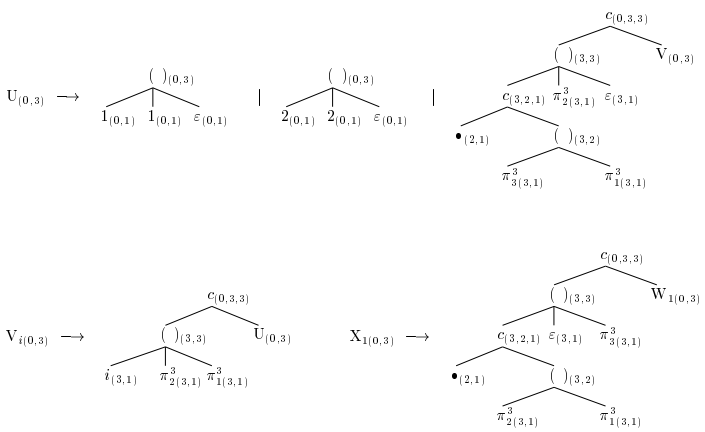
with $F \in \mathsf{F}_{\varepsilon,s}$, and t a term over $T(\Sigma \cup \mathsf{F}_0)$.

An application of a rule $F \longrightarrow t$ "rewrites" a tree rooted in F as the tree t.

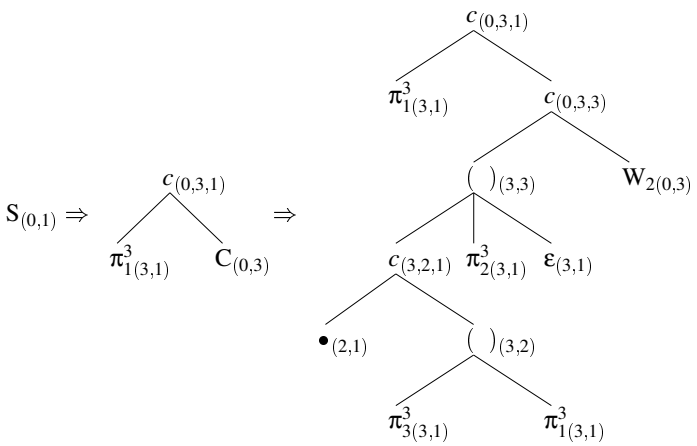
The Translated Example Grammar: $\mathcal{G}'_{ww}$



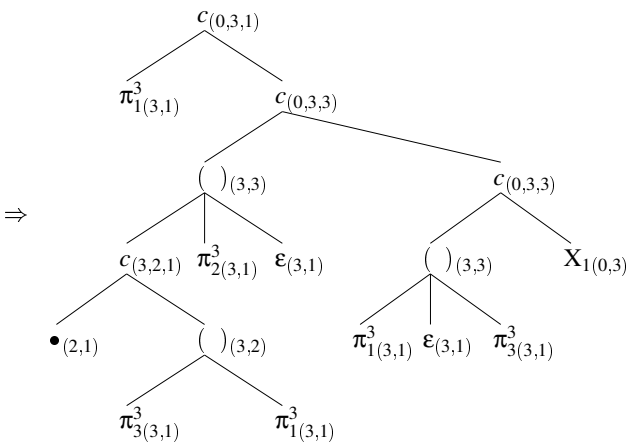
The Translated Example Grammar: $\mathcal{G}'_{ww}$



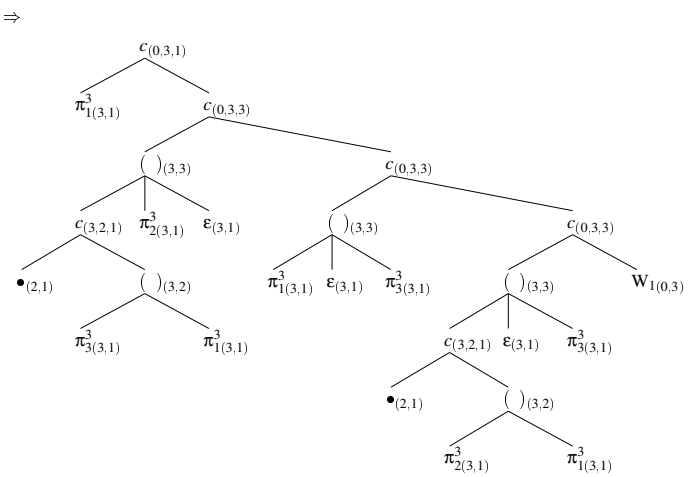
An Example Derivation I



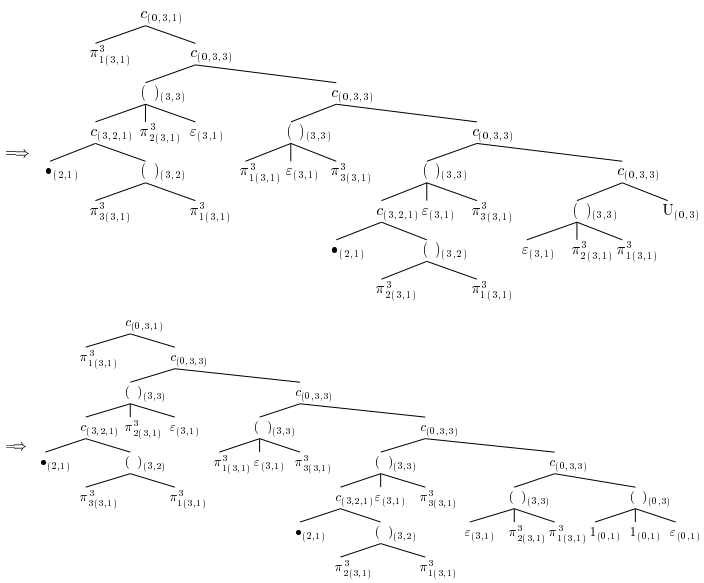
An Example Derivation II



An Example Derivation III



An Example Derivation IV



Coding RTGs in MSO Logic

- Define a tree automaton $\mathfrak{A}_{\mathcal{G}'_{\mathfrak{w}\mathfrak{w}}}$ from the RTG $\mathcal{G}'_{ww}$
- Code its behaviour in $\varphi_{\mathfrak{A}_{\mathcal{G}'_{\mathfrak{w}\mathfrak{w}}}}$

$$
\begin{aligned}
&(\exists X_0,\ldots,X_m)\,[\bigwedge_{i\neq j}(\neg\exists y)\,[y\in X_i\wedge y\in X_j]\wedge\\
&\quad(\forall x)\,[(\neg\exists y)\,[x\lhd y]\rightarrow x\in X_0]\wedge\\
&\quad(\forall x_1,\ldots,x_n,y)[\bigvee_{\substack{(i_1,\ldots,i_n,\sigma,j)\in\alpha\\ 1\leq k\leq n}} x_k\in X_{i_k}\wedge y\lhd x_k\wedge y\in X_j\wedge y\in P_\sigma]\\
&\quad\bigvee_{i\in F}(\exists x\forall y)\,[x\lhd^* y\wedge x\in X_i]
\end{aligned}
$$

Definition (Bottom-Up Tree Automaton).

$$\mathfrak{A} = \langle A, \Sigma, \delta, a_0, F\rangle$$

- A the (finite) set of states,
- Σ a ranked alphabet,
- $a_0 \in A$ the initial state,
- $F \subseteq A$ the final states and
- $\delta : \bigcup_n (A^n \times \Sigma_n) \rightarrow A$ the transition function.

$$
\begin{aligned}
h_\delta(\varepsilon) &= a_0\\
h_\delta(\sigma(t_1,\ldots,t_n)) &= \delta(h_\delta(t_1),\ldots,h_\delta(t_n),\sigma)\\
&\quad t_i\in T_\Sigma, 1\leq i\leq n, \sigma\in\Sigma_n
\end{aligned}
$$

$$T(\mathfrak{A}) = \{t \mid h_\delta(t)\in F\}$$

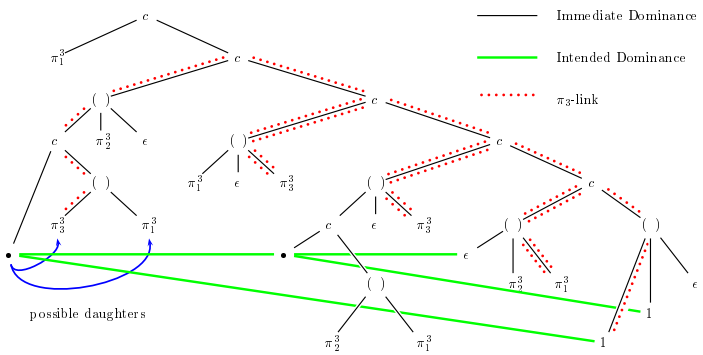
Goal:

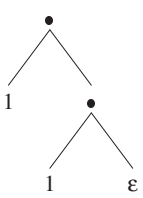
Define a set of relations
$R^I = \{\blacktriangleleft^\star \textit{ (dominance)}, \blacktriangleleft, \blacktriangleleft^+, \lhd \textit{ (precedence)}, \textit{ c-command}, \ldots\}$
holding between the nodes $n \in N^L$ of the explicit tree T^L
which carry a *"linguistic"* label $l \in \mathcal{T}^L \cup \mathcal{N}^L (\subseteq \bigcup_{n\geq 0} \Sigma_n)$
in such a way, that when interpreting

$\blacktriangleleft^\star \in R^I$ as a tree order on the set of *"linguistic"* nodes and
$\lhd \in R^I$ as the precedence relation on the resulting structure,

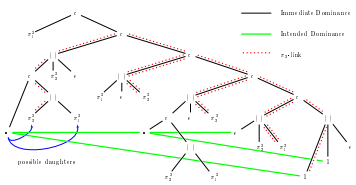
$\langle \{n \mid n \in N^L \wedge \ell(n) \in \mathcal{T}^L \cup \mathcal{N}^L\}, \blacktriangleleft^\star, \lhd \rangle$
is in fact the intended tree corresponding to T^L.

Reconstructing the Intended Structures



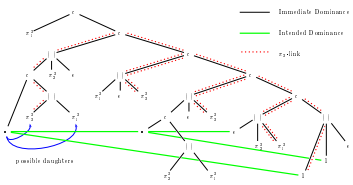


A few general facts



1. Our trees feature three families of labels: the "linguistic" symbols L, i.e., the lifted symbols of the underlying MCFG; the "composition" symbols $\mathrm{C} = \{c_{(u,v,w)}\}$; the "tupling" symbols $(\quad)_{(v,u)}$ and the "projection" symbols $\Pi = \{\pi_i^k\}$.
2. All nonterminal nodes in t' are labeled by some $c \in \mathrm{C}$ or a "tupling" symbol. Note that no terminal node is labeled by some c.
3. The terminal nodes in t' are either labeled by some "linguistic" symbol, a "tupling" symbol of the form $(\quad)_{(k,0)}$, i.e. the "empty" tuple, or by some "projection" symbol π_i^k.

A few facts (cont'd.)



4. Any "linguistic" node dominating anything in the intended tree is on some left branch in t', i.e., it is the left daughter of some $c \in \mathrm{C}$ and the sister of a tupling symbol whose daughters evaluate to the intended daughters.
5. For any node ν labeled with some "projection" symbol $\pi_i^u \in \Pi$ in t' there is a unique node μ (labeled with some $c \in \mathrm{C}$) which properly dominates ν and which properly dominates a node in its other subtree labeled with a "tupling" symbol whose i-th daughter will eventually evaluate to the value of π_i^k. Moreover, μ will be the first node properly dominating ν which is reachable from a leftmost daughter and bears a composition symbol. This crucial fact is arrived at by induction on the construction of $\mathcal{G}'_{ww}$ from $\mathcal{G}_{ww}$.

MSO definable transductions: $\mathcal{R} \rightsquigarrow \mathcal{Q}$

$(\varphi, \psi, (\theta_q)_{q \in Q})$

φ the domain of the transduction

ψ the resulting domain of $\mathcal{Q}$

θ_q the new relations

Definition (Tree-Walking Automaton). A *tree-walking automaton (with tests)* over some ranked alphabet Σ is a finite automaton $\mathfrak{A} = (Q, \Delta, \delta, I, F)$ with states Q, directives Δ, transitions $\delta : Q \times \Delta \to Q$ and the initial and final states $I \subseteq Q$ and $F \subseteq Q$ which traverses a tree using three kinds of directives:

$\uparrow_i$ "move up to the mother of the current node (if it has one and it is its i-th daughter)",

$\downarrow_i$ "move to the i-th daughter of the current node (if it exists)",

$\varphi(x)$ "verify that φ holds at the current node".

For any tree t, such an automaton $\mathfrak{A}$ computes a node relation

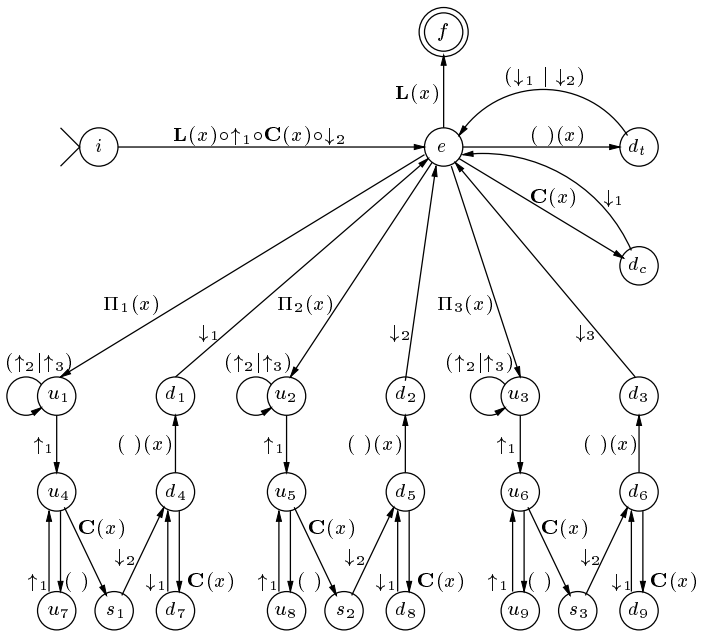
$$R_t(\mathfrak{A}) = \{(x,y) \mid (x,q_i) \stackrel{*}{\Longrightarrow} (y,q_f) \text{ for some } q_i \in I \text{ and some } q_f \in F\}$$

where for all states $q_i, q_j \in Q$ and nodes x, y in t

$$(x,q_i) \Rightarrow (y,q_j) \text{ iff } \exists d \in \Delta : (q_i, d, q_j) \in \delta$$
$$\text{and } y \text{ is reachable from } x \text{ in } t \text{ via } d.$$

If all the tests $\varphi(x)$ of $\mathfrak{A}$ are *MSO* definable, $\mathfrak{A}$ specifies a *regular tree-node relation*, which is itself *MSO* definable (B & E 1997).

The tree-walking automaton A



The walking language $W_{\blacktriangleleft}$

$$W_{\blacktriangleleft} = \mathrm{L}(x) \cdot \uparrow_1 \cdot \mathrm{C}(x) \cdot \downarrow_2 \cdot (W_{(\ )} \cup W_{\mathrm{C}} \cup W_{\Pi_1} \cup W_{\Pi_2} \cup W_{\Pi_3})^* \cdot \mathrm{L}(x)$$

$$\begin{aligned}
W_{(\ )} &= (\ )(x) \cdot (\downarrow_1 \cup \downarrow_2) \\
W_{\mathrm{C}} &= \mathrm{C}(x) \cdot \downarrow_1 \\
W_{\Pi_1} &= \Pi_1(x) \cdot (\uparrow_2 \cup \uparrow_3)^* \cdot \uparrow_1 \cdot ((\ )(x) \cdot \uparrow_1)^* \cdot \mathrm{C}(x) \cdot \downarrow_2 \cdot (\mathrm{C}(x) \cdot \downarrow_1)^* \cdot (\ )(x) \cdot \downarrow_1 \\
W_{\Pi_2} &= \Pi_2(x) \cdot (\uparrow_2 \cup \uparrow_3)^* \cdot \uparrow_1 \cdot ((\ )(x) \cdot \uparrow_1)^* \cdot \mathrm{C}(x) \cdot \downarrow_2 \cdot (\mathrm{C}(x) \cdot \downarrow_1)^* \cdot (\ )(x) \cdot \downarrow_2 \\
W_{\Pi_3} &= \Pi_3(x) \cdot (\uparrow_2 \cup \uparrow_3)^* \cdot \uparrow_1 \cdot ((\ )(x) \cdot \uparrow_1)^* \cdot \mathrm{C}(x) \cdot \downarrow_2 \cdot (\mathrm{C}(x) \cdot \downarrow_1)^* \cdot (\ )(x) \cdot \downarrow_3
\end{aligned}$$

$W_{\blacktriangleleft}$ can be inductively translated into a *MSO*-formula

$$trans_{W_{\blacktriangleleft}}(x,y).$$

Reflexive transitive closure in *MSO* (on finite structures)

$$\textit{R-closed}(X) \equiv (\forall x,y)[x \in X \wedge R(x,y) \rightarrow y \in X]$$
$$[R^*(x,y) \equiv (\forall X)[\textit{R-closed}(X) \wedge x \in X \rightarrow y \in X]$$

Walking language $\Rightarrow$ *MSO*-formula

$$\begin{aligned}
trans_{\emptyset}(x,y) &= \bot \\
trans_{\downarrow_i}(x,y) &= edge_i(x,y) \\
trans_{\uparrow_i}(x,y) &= edge_i(y,x) \\
trans_{\varphi(x)}(x,y) &= \varphi(x) \wedge x = y \\
trans_{W_1 \cup W_2}(x,y) &= trans_{W_1}(x,y) \vee trans_{W_2}(x,y) \\
trans_{W_1 \cdot W_2}(x,y) &= \exists z\,(trans_{W_1}(x,z) \wedge trans_{W_2}(z,y)) \\
trans_{W^*}(x,y) &= trans^*_W(x,y) \\
trans^*_W(x,y) &= \forall X\,(\forall v,w\,(v \in X \wedge trans_W(v,w) \rightarrow w \in X) \\
&\qquad \wedge x \in X \rightarrow y \in X)
\end{aligned}$$

Intended Structures via MSO transduction

$$(\varphi, \psi, (\theta_q)_{q \in Q})$$

$$Q = \{\blacktriangleleft, \blacktriangleleft^\star, \blacktriangleleft^+, \triangleleft, \dots\}$$

$$\begin{aligned}
\varphi &= \text{ the formula coding } \mathfrak{A}_{\mathcal{G}'_{ww}} \\
\psi &= (\exists y)[\mathsf{trans}_{W_{\blacktriangleleft}}(x,y) \vee \mathsf{trans}_{W_{\blacktriangleleft}}(y,x)] \\
\theta_{\blacktriangleleft} &= \mathsf{trans}_{W_{\blacktriangleleft}}(x,y) \\
\theta_{\blacktriangleleft^\star} &= (\forall X)[\blacktriangleleft\textit{-closed}(X) \vee x \in X \rightarrow y \in X] \\
\theta_{\blacktriangleleft^+} &= x \blacktriangleleft^\star y \vee x \not\approx y \\
\theta_{\triangleleft} &= \text{ another tree-walking automaton} \\
\theta_{\text{labels}} &= \text{ as in the old structure } R
\end{aligned}$$

Outlook

Theorem (Courcelle). The image of an MSO-definable class of structures under a definable transduction is not MSO definable in general.

But which class of structures is captured using regular means/MSO logic combined with regular/MSO definable binary node relations?

Is there a machine model for the MSO transduction such that we can implement the tree translation?