Combination of Constraint Systems II: Rational Amalgamation ⋆

Klaus U. Schulz

CIS, Universität München, Oettingenstr. 67, 80538 München, Germany
e-mail: schulz@cis.uni-muenchen.de

Stephan Kepser

SfS, Universität Tübingen, Wilhelmstr. 113, 72074 Tübingen, Germany
e-mail: kepser@sfs.nphil.uni-tuebingen.de

Abstract

In two earlier papers [1], the concept of "free amalgamation" has been introduced as a general methodology for interweaving solution structures for symbolic constraints, and it was shown how constraint solvers for two components can be lifted to a constraint solver for the free amalgam. Here we discuss a second general way for combining solution domains, called *rational amalgamation*. In praxis, rational amalgamation seems to be the preferred combination principle if the two solution structures to be combined are "rational" or "non-wellfounded" domains. It represents, e.g., the way how rational trees and rational lists are interwoven in the solution domain of Prolog III, and a variant has been used by W. Rounds for combining feature structures and hereditarily finite non-wellfounded sets. We show that rational amalgamation is a general combination principle, applicable to a large class of structures. As in the case of free amalgamation, constraint solvers for two component structures can be combined to a constraint solver for their rational amalgam. From this algorithmic point of view, rational amalgamation seems to be interesting since the combination technique for rational amalgamation avoids one source of non-determinism that is needed in the corresponding scheme for free amalgamation.

[1] see [4,5].

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1 Introduction

One idea behind constraint solving is to use specialized formalisms and inference mechanisms to solve domain-specific tasks. In many applications, however, one is faced with a complex combination of different problems, which means that a system tailored to solving a single problem can only be applied if it is possible to combine it both with other specialized systems and with general purpose systems. The present paper, as its predecessors [4,5], marks one step in a program where we try to characterize the most important *general constructions* for combining solution domains and constraint solvers for symbolic constraints.

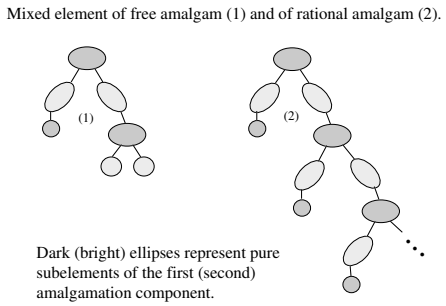
A general combination method, in our sense, has to give answers to two problems. First, it must offer a general construction for combining two solution domains. Second, a combination algorithm has to be given that reduces the problem of solving "mixed" constraints over the combined solution domain to the problem of solving "pure" constraints over the two component structures. We think that it is in fact possible to characterize a small set of fundamental combination methods that describe—modulo minor deviations—all known instances of combined symbolic constraint systems in this sense.

In [4,5] the notion of the *free amalgamated product* of two component structures was introduced. This product is characterized by a universality-property: it represents a most general object among all structures that can be considered as a reasonable combination of the two components, in the sense that each such combination is a homomorphic image of the free amalgamated product. For so-called SC-structures[2] over disjoint signatures an explicit construction of the free amalgamated product of two components was given and it was shown how given constraint solvers for the component structures can be combined to a constraint solver for the free amalgam.

In the present paper we introduce a second systematic way to combine constraint systems over SC-structures, called *rational amalgamation*. Free and rational amalgamation both yield a combined structure with "mixed" elements that interweave a finite number of "pure" elements of the two components in a particular way. The difference between both constructions becomes transparent when we ignore the interior structure of these pure subelements and consider them as construction units with a fixed arity, similar to "complex function symbols". Under this perspective, and ignoring details that concern the ordering of the children of a node, mixed elements of the free amalgam can be considered as finite trees, whereas mixed elements of the rational amalgam

[2] SC-structures (simply combinable structures) have been introduced in [4]. Later, in [5], the name "quasi-free structure" was used for the restricted class of *strong* SC-structures, cf. Def. 3.3.

are like rational trees [3] .



On this background it should not be surprising that in praxis rational amalgamation appears to be the preferred combination principle in situations where the two solution structures to be combined are themselves "rational" or "cyclic" domains: for example, it represents the way how rational trees and rational lists are interwoven in the solution domain of Prolog III ([9]), and a variant of rational amalgamation has been used to combine feature structures with non-wellfounded sets in a system introduced by W. Rounds [17].

We introduce rational amalgamation as a general construction that can be used to combine so-called non-collapsing SC-structures over disjoint signatures. If the symbols of the signature of one component are ignored, the rational amalgamated product is isomorphic to the remaining component. It is shown how constraint solving in the rational amalgam can be reduced to constraint solving in the components. The decomposition scheme that is used is closely related to the decomposition algorithm for free amalgamation, but it avoids one highly non-deterministic step that is needed in the latter scheme. Hence, when matters of efficiency become important, rational amalgamation might be the better choice.

Let us now briefly indicate which insights could be gained from a classification of the basic methodologies for combining constraints systems. Below we shall summarize what has been obtained so far.

(1) It helps to understand the scale of possibilities and the general limitations for combining constraints systems.
(2) It might facilitate the design of new combined constraint systems, and it helps to understand existing instances of combination from a general point of view.
(3) It establishes new and interesting connections between the theory of constraint solving and other areas such as, e.g., universal algebra and logic.
(4) The relationship between different methodologies for combining constraint systems is interesting per se, we hope to verify.

[3] A possibly infinite tree is *rational* if it is finitely branching and has only a finite number of distinct subtrees. See [8,13,10].

1. From our present perspective, which is explained in more detail in the conclusion, free and rational amalgamation, and a related construction called "infinite amalgamation" seem to be the most important combination principles in a spectrum of related methods. Furthermore, we are confident that the abstract definition of an SC-structure, as introduced in [4] and used here, captures a maximal class of (unsorted!) structures where these combination principles can be applied in a uniform way. This class covers most of the non-numerical and non-finite solution domains that are used in constraint solving. More specifically, all the solution domains that are considered in the area of unification modulo equational theories are SC-structures. Furthermore, the algebra of rational trees, feature structures, and structures with finite or rational nested sets, lists and multisets are SC-structures.

2. The results of this paper show, e.g., that there is a common and general methodology behind Colmerauer's combination of rational trees and rational lists in the solution domain of Prolog III ([9]) and Rounds' [17] combination of feature structures with non-wellfounded sets. The amalgamation technique to be described in the present paper can be used, e.g., to obtain similar combinations where rational trees, feature structures, rational lists, nested multi-sets, or quotient term algebras for collapse-free equational theories over disjoint signatures are interweaved in arbitrary manner.

3. The purely algebraic definition of an SC-structure directly generalizes the notion of a free structure (see [5] for a thorough discussion). Still, SC-structures have what is sometimes called the "universal mapping property" of free structures, and a major part of the theory of free structures as developed in universal algebra can be lifted to the case of SC-structures. A detailed mathematical investigation of this point is in progress. Furthermore, it has turned out that the methods for combining solution domains developed in [4] and here, and the general methods for combining logics described by Gabbay [11] and Pfalzgraf [15,16] follow the very same abstract idea. See [11] for a first discussion of this issue.

4. One interesting connection between free and rational amalgamation relies on the observation that the free amalgamated product is always a substructure of the rational amalgamated product. The conclusion will be used to comment on item 4 in more detail.

As it should have become clear from the introduction, the present paper tries to obtain results of maximal generality. This makes it inevitable to use a high level of abstraction. The following section summarizes the algebraic concepts that are needed subsequently.

2 Preliminaries

A signature Σ consists of a set Σ_F of function symbols and a disjoint set Σ_P of predicate symbols (not containing "="), each of fixed arity. The signatures considered in this paper may be finite or countably infinite. Expressions $\mathcal{A}^\Sigma$ denote Σ-structures over the carrier set A, and $f_\mathcal{A}$ ($p_\mathcal{A}$) stands for the interpretation of $f \in \Sigma_F$ ($p \in \Sigma_P$) in $\mathcal{A}^\Sigma$. Σ-terms ($t, t_1, \ldots$) and atomic Σ-formulas (of the form $t_1 = t_2$, or of the form $p(t_1, \ldots, t_n)$) are built as usual. A Σ-formula φ is written in the form $\varphi(v_1, \ldots, v_n)$ in order to indicate that the free variables of φ are in $\{v_1, \ldots, v_n\}$. We write $\mathcal{A}^\Sigma \models \varphi(v_1/a_1, \ldots, v_n/a_n)$ if φ becomes true in $\mathcal{A}^\Sigma$ under all assignments that map v_i to $a_i \in A$, for $1 \leq i \leq n$.

A Σ-homomorphism is a mapping h between two structures $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Sigma$ such that $h(f_\mathcal{A}(a_1, \ldots, a_n)) = f_\mathcal{B}(h(a_1), \ldots, h(a_n))$ and $p_\mathcal{A}[a_1, \ldots, a_n]$ implies that $p_\mathcal{B}[h(a_1), \ldots, h(a_n)]$ for all $f \in \Sigma_F$, $p \in \Sigma_P$, and $a_1, \ldots, a_n \in A$. A Σ-isomorphism is a bijective Σ-homomorphism $h : \mathcal{A}^\Sigma \to \mathcal{B}^\Sigma$ such that $p_\mathcal{A}[a_1, \ldots, a_n]$ if, and only if, $p_\mathcal{B}[h(a_1), \ldots, h(a_n)]$, for all $a_1, \ldots, a_n \in A$. A Σ-endomorphism of $\mathcal{A}^\Sigma$ is a homomorphism $h^\Sigma : \mathcal{A}^\Sigma \to \mathcal{A}^\Sigma$. With $End^\Sigma_\mathcal{A}$ we denote the monoid of all endomorphisms of $\mathcal{A}^\Sigma$, with composition as operation. A Σ-endomorphism which is an isomorphism is a Σ-automorphism.

Mappings, as usual, are formally treated as sets of ordered pairs. If $g : A \to B$ and $h : B \to C$ are mappings, then $g \circ h : A \to C$ denotes their composition. Expressions like $\vec{v}, \vec{a}$ are used to denote finite sequences. If $\vec{a} = a_1, \ldots, a_n$ is a sequence of elements of A and if m is a mapping with domain A, then $m(\vec{a})$ denotes the sequence $m(a_1), \ldots, m(a_n)$. If $\vec{v} = v_1, \ldots, v_n$, then $\mathcal{A}^\Sigma \models \varphi(\vec{v}/\vec{a})$ is shorthand for $\mathcal{A}^\Sigma \models \varphi(v_1/a_1, \ldots, v_n/a_n)$. The symbol "$\uplus$" denotes disjoint set union. With $|A|$ we denote the cardinality of the set A.

3 Non-collapsing SC-structures

In this section we introduce the class of structures for which we can use the rational amalgamation construction (Definition 3.11). First we recall the definition of an SC-structure given in [4]. The motivation to introduce this class was the observation that most of the non-numerical and non-finite solution domains that are used in different areas of constraint programming can be treated on a common algebraic background when we generalize the concept of a free structure appropriately. To help the reader, the algebra of rational trees will be used to exemplify the following abstract concepts. In the sequel, we consider a fixed Σ-structure $\mathcal{A}^\Sigma$, and $\mathcal{M}$ denotes a submonoid of $End^\Sigma_\mathcal{A}$.

Definition 3.1 Let A_0, A_1 be subsets of $\mathcal{A}^\Sigma$. Then A_0 *stabilizes* A_1 *with respect to* $\mathcal{M}$ iff all elements m_1 and m_2 of $\mathcal{M}$ that coincide on A_0 also coincide

on A_1. For $A_0 \subseteq A$ the *stable hull of* A_0 *with respect to* $\mathcal{M}$ is the set

$$SH^{\mathcal{A}}_{\mathcal{M}}(A_0) := \{a \in A \mid A_0 \text{ stabilizes } \{a\} \text{ with respect to } \mathcal{M}\}.$$

The stable hull of a set A_0 has properties that are similar to those of the subalgebra generated by A_0: $SH^{\mathcal{A}}_{\mathcal{M}}(A_0)$ is always a Σ-substructure of $\mathcal{A}^\Sigma$, and $A_0 \subseteq SH^{\mathcal{A}}_{\mathcal{M}}(A_0)$. In general, however, the stable hull can be larger than the generated subalgebra. For example, if $\mathcal{A}^\Sigma := R(\Sigma, X)$ denotes the algebra of rational trees over signature Σ, if $\mathcal{M} = End^\Sigma_{\mathcal{A}}$, and if $Y \subseteq X$ is a subset of the set of variables, X, then $SH^{\mathcal{A}}_{\mathcal{M}}(Y)$ contains all *rational* trees with variables in Y, while Y generates all *finite* trees with variables in Y only.

Definition 3.2 The set $X \subseteq A$ is an $\mathcal{M}$*-atom set for* $\mathcal{A}^\Sigma$ if every mapping $X \to A$ can be extended to an endomorphism in $\mathcal{M}$. If $\mathcal{M} = End^\Sigma_{\mathcal{A}}$, or if $\mathcal{M}$ is clear from the context, then X is simply called an *atom set* for $\mathcal{A}^\Sigma$.

For example, if $\mathcal{A}^\Sigma := R(\Sigma, X)$ is the algebra of rational trees over the set of variables X, then X is an atom set for $\mathcal{A}^\Sigma$.

Definition 3.3 A countably infinite Σ-structure $\mathcal{A}^\Sigma$ is an *SC-structure* (simply combinable structure) iff there exists a submonoid $\mathcal{M}$ of $End^\Sigma_{\mathcal{A}}$ such that $\mathcal{A}^\Sigma$ has an infinite $\mathcal{M}$-atom set X where every $a \in A$ is stabilized by a finite subset of X with respect to $\mathcal{M}$. We denote this SC-structure by $(\mathcal{A}^\Sigma, \mathcal{M}, X)$. If $\mathcal{M} = End^\Sigma_{\mathcal{A}}$, then $(\mathcal{A}^\Sigma, End^\Sigma_{\mathcal{A}}, X)$ is called a *strong* SC-structure [4] .

Examples 3.4 Each quotient $T(\Sigma, X)/=_E$ of a term algebra $T(\Sigma, X)$ modulo an equational theory E, free structures, rational tree algebras, as well as nested, hereditarily finite wellfounded or non-wellfounded sets, multisets, and lists, and various types of feature structures are SC-structures. In each case we have to assume the presence of a countably infinite set of atoms (variables, urelements, etc.). See [5] for formal details.

Let us remark that the restriction to countably infinite structures in Definition 3.3 is not necessarily inevitable. For the following considerations, just three properties of SC-structures $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ are important: first, $|X|$, has to be infinite. Second, we need that $|X| = |A|$. Third, the cardinality of (suitable) sets that stabilize the elements $a \in A$ has to be smaller than $|X|$. If we use these three conditions as a starting point in Definition 3.3 and try to derive the corresponding results, proofs will become more involved. Since uncountable structures seem to be not really relevant in our context, we decided to avoid these additional difficulties.

In the rest of this section, $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ denotes a fixed SC-structure with carrier A.

[4] Strong SC-structures have been called *quasi-free structures* in later publications, e.g. ([5].)

Lemma 3.5 *Let $\varphi(v_1, \ldots, v_k)$ be a positive Σ-formula, let $m \in \mathcal{M}$, and let $a_1, \ldots, a_k$ be elements of A. Then $\mathcal{A}^\Sigma \models \varphi(v_1/a_1, \ldots, v_k/a_k)$ implies $\mathcal{A}^\Sigma \models \varphi(v_1/m(a_1), \ldots, v_k/m(a_k))$.*

Proof. It is simple to see that there exists a surjective endomorphism $m' \in \mathcal{M}$ that coincides with m on $\{a_1, \ldots, a_k\}$. The result follows from the well-known fact that validity of positive formulae is preserved under surjective homomorphisms. □

Lemma 3.6 *Let $\varphi(v_1, \ldots, v_k)$ be a positive Σ-formula, let $x_1, \ldots, x_k$ be distinct atoms in X. Then $\mathcal{A}^\Sigma \models \varphi(v_1/x_1, \ldots, v_k/x_k)$ implies that $\mathcal{A}^\Sigma \models \forall v_1, \ldots \forall v_k\ \varphi$.*

Proof. Let $a_1, \ldots, a_k$ be arbitrary elements of A. Since X is an $\mathcal{M}$-atom set, there exists an $m \in \mathcal{M}$ such that $m(x_i) = a_i$, for $i = 1, \ldots, k$. Hence $\mathcal{A}^\Sigma \models \varphi(v_1/x_1, \ldots, v_k/x_k)$ implies $\mathcal{A}^\Sigma \models \varphi(v_1/a_1, \ldots, v_k/a_k)$, by Lemma 3.5. It follows that $\mathcal{A}^\Sigma \models \forall v_1 \ldots \forall v_k\ \varphi$. □

A fundamental property of SC-structures is the following (cf. [4], Lemma 13).

Lemma 3.7 *For each $a \in A$ there exists a* unique minimal *finite subset Y of X such that $a \in SH^{\mathcal{A}}_{\mathcal{M}}(Y)$.*

Definition 3.8 The *stabilizer of $a \in A$ with respect to $\mathcal{M}$* [5], $Stab^{\mathcal{A}}_{\mathcal{M}}(a)$ is the unique minimal finite subset Y of X such that $a \in SH^{\mathcal{A}}_{\mathcal{M}}(Y)$. The stabilizer of $A' \subseteq A$ is the set $Stab^{\mathcal{A}}_{\mathcal{M}}(A') := \bigcup_{a \in A'} Stab^{\mathcal{A}}_{\mathcal{M}}(a)$.

For the mathematical treatment of SC-structures, the concept of the stabilizer turns out to be extremely useful. It might give a good intuition to imagine that the stabilizer of an element a is the set of atoms "occurring" in a. If a is an element of an algebra of rational trees over the set of variables V, then the stabilizer of a is in fact the set of variables occurring in the rational tree a. Note, however, that "the" set of atoms (variables) occurring, e.g., in distinct terms that represent the same element of a quotient term algebra is *not* unique in general. It is trivial to see that $SH^{\mathcal{A}}_{\mathcal{M}}(Y) = \{a \in A \mid Stab^{\mathcal{A}}_{\mathcal{M}}(a) \subseteq Y\}$, for each $Y \subseteq X$. In the sequel, further properties of stabilizers will be used. The first lemma is a trivial consequence of the fact that stable hulls are Σ-substructures.

Lemma 3.9 *Let $f \in \Sigma$ be an n-place operator and $a_1, \ldots, a_n \in A$. Then $Stab^{\mathcal{A}}_{\mathcal{M}}(f_{\mathcal{A}}(a_1, \ldots, a_n)) \subseteq Stab^{\mathcal{A}}_{\mathcal{M}}(\{a_1, \ldots a_n\})$.*

The next lemma plays a crucial role in the rational amalgamation construction. It will be used in many proofs.

[5] Whenever the monoid $\mathcal{M}$ is clear from the context, we shall not mention it.

Lemma 3.10 *Let $m \in \mathcal{M}$ be an endomorphism of the SC-structure $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ such that the restriction of m on X is a mapping $X \to X$. If $Stab^{\mathcal{A}}_{\mathcal{M}}(a) = \{x_1, \ldots, x_k\}$, then $Stab^{\mathcal{A}}_{\mathcal{M}}(m(a)) \subseteq \{m(x_1), \ldots, m(x_k)\}$. If m is an automorphism, then $Stab^{\mathcal{A}}_{\mathcal{M}}(m(a)) = \{m(x_1), \ldots, m(x_k)\}$.*

Proof. Let m_1 and m_2 be two endomorphisms in $\mathcal{M}$ that coincide on $\{m(x_1), \ldots, m(x_k)\} \subseteq X$. Then $m \circ m_1$ and $m \circ m_2$ are two endomorphisms in $\mathcal{M}$ that coincide on $\{x_1, \ldots, x_k\}$. By assumption, $m \circ m_1$ and $m \circ m_2$ coincide on a. But then m_1 and m_2 coincide on $m(a)$. Hence $Stab^{\mathcal{A}}_{\mathcal{M}}(m(a)) \subseteq \{m(x_1), \ldots, m(x_k)\}$. Assume that m is an automorphism, and that $Stab^{\mathcal{A}}_{\mathcal{M}}(m(a))$ is a proper subset of $\{m(x_1), \ldots, m(x_k)\}$. The first part of the lemma, applied to m^{-1}, yields a proper subset of $\{x_1, \ldots, x_k\}$ that stabilizes a, which is impossible, by choice of $\{x_1, \ldots, x_k\}$. □

We may now characterize the subclass of SC-structures for which we can use the rational amalgamation construction.

Definition 3.11 An SC-structure $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ is *non-collapsing* if every endomorphism $m \in \mathcal{M}$ maps non-atoms to non-atoms (i.e., $m(a) \in A \setminus X$ for all $a \in A \setminus X$ and all $m \in \mathcal{M}$).

For example, quotient term algebras for collapse-free equational theories, rational tree algebras, feature structures, feature structures with arity, the domains with nested, finite or rational lists, and the domains with nested, finite or rational multi-sets (as mentioned in 3.4) are always non-collapsing.

Let us note that the domains with nested, finite or rational sets do not belong to the class of non-collapsing SC-structures. The reason is that in this case atoms have the form $\{y\}$, where y is taken from a countably infinite set of urelements Y. Since we do not use sorts, and since union of urelements is not defined, the urelements itself do not belong to the structure. If y_1 and y_2 are distinct urelements, then $\{y_1, y_2\} = \{y_1\} \cup \{y_2\}$ is a non-atomic element. Now any endomorphism that maps the atom $\{y_1\}$ to $\{y_2\}$ and leaves $\{y_2\}$ fixed "collapses" the non-atom $\{y_1, y_2\}$ to the atom $\{y_2\}$.[6]

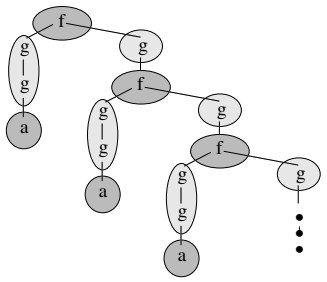
4 The Rational Amalgamated Product

In this section we define the rational amalgamated product of two non-collapsing SC-structures over disjoint signatures. We proceed in two steps. In the first subsection we describe the underlying domain of the rational amalgam. In the second subsection we introduce functions and relations.

[6] We think that this unpleasant effect disappears when we use sorts. With sorts, it should be possible to use the set of urelements as atom set. Of course $\{y_1, y_2\}$ may still be mapped to $\{y_2\}$, but the latter is a non-atomic element now.

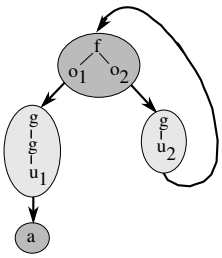
With the rational amalgamation construction we would like to interweave elements of two distinct structures in a regular way. We want to obtain a general construction where the definition of mixed elements does not depend on the particular form of the elements of the two components. The following example will serve as a motivation for our formalization of combined elements.

Example 4.1 Let $\mathcal{A}^\Sigma$ (resp. $\mathcal{B}^\Delta$) denote the algebra all finite, ordered trees over the signature $\Sigma := \{f, a\}$ (resp. $\Delta := \{g\}$), leaves possibly labelled with variables. The following figure represents one element t of the intended combined domain.



We may look at t as an ordered rational tree where nodes are labelled with function symbols from the joint signature $\Sigma \cup \Delta$. However, it is unclear how such a perspective can be generalized to more general components $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$. For example, a function $f_\mathcal{A}$ may satisfy semantic properties such as commutativity or associativity. Since the structure of the combined domain should reflect the structure of the two components as closely as possible, the use of ordered rational trees for representing mixed elements would be inappropriate in this case.

From another point of view, the mixed tree t can be considered as an $\mathcal{A}^\Sigma$-context $f_\mathcal{A}(.,.) = f(.,.)$ that is applied to $\mathcal{B}^\Delta$-contexts $g(g(.))$ and $g(.)$, which in turn are applied to $\mathcal{A}^\Sigma$-terms/contexts a and $f(.,.)$ respectively etc. In order to formalize an $\mathcal{A}^\Sigma$-context $f(.,.)$ we may use an element $f(o_1, o_2)$ of $\mathcal{A}^\Sigma$ where o_1, o_2 are special atoms of $\mathcal{A}^\Sigma$ that play the role of "holes". In this way, each element of the combined domain is composed of a finite number of elements of the two components. As we shall see, in situations where more general component structures are considered this guarantees that semantic properties of function symbols are automatically taken into account. For modelling application of a context $f(.,.)$ to arguments $g(g(.))$ and $g(.)$ we use "fibering functions" that map the atoms o_1 and o_2 to $g(g(u_1))$ and $g(u_2)$ respectively, where the latter terms denote the elements of $\mathcal{B}^\Delta$ that represent $g(g(.))$ and $g(.)$. We arrive at the following modified representation of t.



It should be clear that a formalization of mixed elements that just uses the elements of the two components plus a fibering function can immediately be generalized to arbitrary SC-structures. Below, an object that interweaves a finite number of elements of given component structures by means of a fibering function will be called a braid. We shall use braids for representing the elements of the rational amalgam. The main problem with this approach is the fact that different braids may represent the same object, as we shall see below. For this reason, results are given that eventually show that we may define a unique normal form for each braid. For the sake of readability, the rather technical proofs of these results are shifted to a separate section (Section 8).

In the sequel, we describe the rational amalgamation of two component structures. The same construction may be used to interweave an arbitrary number of component structures. Throughout this section $(\mathcal{A}^{\Sigma}, \mathcal{M}, X)$ and $(\mathcal{B}^{\Delta}, \mathcal{N}, Y)$ denote two fixed non-collapsing SC-structures over disjoint signatures. We assume that the atom sets X and Y have the form $X = Z \uplus \mathcal{O}_A$ and $Y = Z \uplus \mathcal{O}_B$, where the sets $Z, \mathcal{O}_A$, and $\mathcal{O}_B$ are all infinite, and where $\mathcal{O}_A \cap \mathcal{O}_B = \emptyset$. The atoms in Z will be called *bottom atoms*, the atoms in $\mathcal{O}_A$ ($\mathcal{O}_B$) will be called *open atoms*. Bottom atoms will play the role of ordinary atoms, or leaves. Open atoms, in contrast, are used to fiber elements of both structures. With $\mathcal{O}_A(a)$ and $\mathcal{O}_A(A')$ we denote the set of open atoms occurring in the stabilizer of $a \in A$ ($A' \subseteq A$) with respect to $\mathcal{M}$. Similarly expressions $\mathcal{O}_B(b)$ ($\mathcal{O}_B(B')$) are used to denote the set of open atoms occurring in the stabilizer of $b \in B$ ($B' \subseteq B$) with respect to $\mathcal{N}$.

Definition 4.2 Let $\mathcal{O}'_A \subseteq \mathcal{O}_A$. A *fibering function* for $\mathcal{O}'_A$ is a mapping $\pi_A : \mathcal{O}'_A \to B \setminus Y$. Fibering functions for subsets $\mathcal{O}'_B \subseteq \mathcal{O}_B$ are defined symmetrically and assign non-atomic elements of A to the open atoms in $\mathcal{O}'_B$.

Definition 4.3 Let $\mathcal{O}'_A \subseteq \mathcal{O}_A$, $\mathcal{O}'_B \subseteq \mathcal{O}_B$, let π_A and π_B be fibering functions for $\mathcal{O}'_A$ and $\mathcal{O}'_A$ respectively, let $\pi := \pi_A \cup \pi_B$. An element $a \in A$ is *directly fibered* with $b \in B$ via π if there is an $o \in \mathcal{O}_B(b)$ such that $a = \pi_B(o)$. Symmetrically $b \in B$ is *directly fibered* with $a \in A$ via π if there exists an $o \in \mathcal{O}_A(a)$ such that $b = \pi_A(o)$. An element $a \in A \cup B$ is *fibered* with $b \in A \cup B$ via π if there exists a sequence $a = a_0, a_1, \ldots, a_n = b$ ($n \geq 0$) such that each a_i is directly fibered with a_{i+1} via π, for $0 \leq i \leq n-1$.

We can now introduce the main concept of this section.

Definition 4.4 A *braid* of type A over $\mathcal{A}^\Sigma, \mathcal{B}^\Delta$ is a quintuple $\mathcal{K} = \langle a, C, D, \pi_A, \pi_B \rangle$, where

(1) $a \in A \setminus \mathcal{O}_A$,
(2) C is a finite subset of A containing a. All elements of $C \setminus \{a\}$ are *non-atomic*. D is a finite set of *non-atomic* elements of B,
(3) $\pi_A : \mathcal{O}_A(C) \to D$ and $\pi_B : \mathcal{O}_B(D) \to C$ are fibering functions,
(4) each element in $C \cup D$ fibered with a via $\pi := \pi_A \cup \pi_B$.

The element a is called the *root* of $\mathcal{K}$. The elements in the sets C and D are called the *elements of $\mathcal{K}$ of type A and B* respectively. The functions π_A and π_B are called the *fibering functions* of $\mathcal{K}$. We sometimes write $\mathcal{O}_A(\mathcal{K})$ and $\mathcal{O}_B(\mathcal{K})$ for $\mathcal{O}_A(C)$ and $\mathcal{O}_B(D)$ respectively, and $\mathcal{O}(\mathcal{K})$ denotes the union $\mathcal{O}_A(\mathcal{K}) \cup \mathcal{O}_B(\mathcal{K})$.

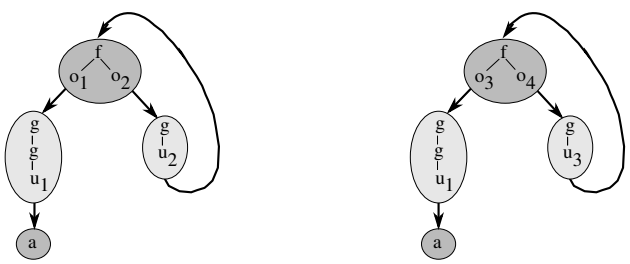
Braids of type B, with root in $B \setminus \mathcal{O}_B$, are defined symmetrically. A braid $\mathcal{K}$ is called *trivial* if the root of $\mathcal{K}$ is a bottom atom $z \in Z$. In this case, z is the only element of the braid. It does not make sense to distinguish between the trivial braid $\langle z, \{z\}, \emptyset, \emptyset, \emptyset \rangle$ of type A and the trivial braid $\langle z, \emptyset, \{z\}, \emptyset, \emptyset \rangle$ of type B. We identify both braids. Hence, trivial braids have mixed type.

It is sometimes convenient to consider quintuples of the form $\langle a, C, D, \pi_A, \pi_B \rangle$ that satisfy Conditions 1-3 above (but not necessarily Condition 4). Such a quintuple will be called a *braid forest*.

Definition 4.5 A braid (forest) $\mathcal{K}_2$ is called a *subbraid* (*subforest*) of a braid (forest) $\mathcal{K}_1$ if each element of $\mathcal{K}_2$ is an element of $\mathcal{K}_1$ and if the fibering function of $\mathcal{K}_2$ (considered as a set of ordered pairs) is a subset of the fibering function of $\mathcal{K}_1$.

In Section 8 (cf. Lemma 8.4) we show that for each element e of a braid (forest) $\mathcal{K}$ there exists a unique subbraid of $\mathcal{K}$ with root e.

Example 4.6 The following figure depicts two braids $\mathcal{K}_1$ and $\mathcal{K}_2$ over two free term algebras, for signatures $\Sigma = \{f, a\}$ and $\Delta = \{g\}$ respectively. The roots are the topmost elements $f(o_1, o_2)$ and $f(o_3, o_4)$ respectively. The elements $g(g(u_1))$ and a build a common subbraid of both braids, with fibering function $\{\langle u_1, a \rangle\}$.



The discussion in Example 4.1 shows that both braids are meant to denote the same object since they represent the same rational tree. More generally we would like to identify two braids that are identical modulo a bijective renaming of open atoms. This motivates the following definitions.

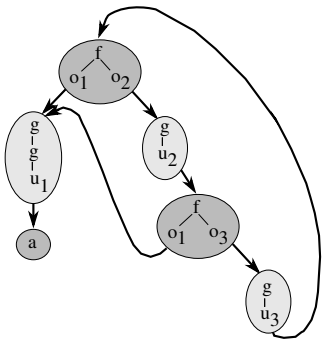
An endomorphism $m \in \mathcal{M}$ ($n \in \mathcal{N}$) is called *admissible* if m (n) leaves all bottom atoms $z \in Z$ fixed and if $m(o) \in \mathcal{O}_A$ ($n(o) \in \mathcal{O}_B$) for all $o \in \mathcal{O}_A$ ($o \in \mathcal{O}_B$).[7] A pair $(m, n) \in \mathcal{M} \times \mathcal{N}$ is called admissible if both m and n are admissible.

Definition 4.7 Let $\mathcal{K} = \langle a, C, D, \pi_A, \pi_B \rangle$ and $\mathcal{K}' = \langle a', C', D', \pi'_A, \pi'_B \rangle$ be two braids (braid forests), say, of type A. $\mathcal{K}'$ is called a *variant* of $\mathcal{K}$ if there exists an admisible pair of automorphisms (m, n) such that

(1) $a' = m(a)$,
(2) $C' = \{m(c) \mid c \in C\}$, and $D' = \{n(d) \mid d \in D\}$,
(3) $\pi'_A := \{\langle m(o), n(d) \rangle \mid \langle o, d \rangle \in \pi_A\}$, and
$\pi'_B := \{\langle n(o), m(c) \rangle \mid \langle o, c \rangle \in \pi_B\}$.

We write $\mathcal{K}_1 \sim \mathcal{K}_2$ if $\mathcal{K}_1$ and $\mathcal{K}_2$ are variants. Since the set of admissible automorphisms of $\mathcal{A}^\Sigma$ (resp. $\mathcal{B}^\Delta$) defines (w.r.t. composition) a group, "$\sim$" defines an equivalence relation on the set of all braids (braid forests) for $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$.

Unfortunately, different braids may represent the same object even if they are not variants. As an example, consider the following braid $\mathcal{K}_3$.



It represents the same object as the braids $\mathcal{K}_1$ and $\mathcal{K}_2$ given in our previous figures since it stands for the same rational tree. In order to capture this relationship we shall now define the concept of simplification of braids. Basically, the idea behind simplification is to "identify" open atoms that point to subbraids representing the same object. For example, in $\mathcal{K}_3$ the atoms o_2 and o_3 point to the subbraids with roots $g(u_2)$ and $g(u_3)$ respectively, and these two subbraids represent the same rational tree. In order to identify o_2 and o_3 we

[7] Intuitively, admissible endomorphisms cause a "renaming" of open atoms, compare Lemma 3.10. They may identify distinct open atoms.

use an endomorphism $m \in \mathcal{M}$ that maps o_3 to o_2 and leaves o_1, o_2 fixed. When we apply m to the elements of type A of $\mathcal{K}_3$, we invalidate the definition of a braid, since the open atom o_2 is now fibered with two elements, $g(u_2)$ and $g(u_3)$. We obtain a proper fibering function when we identify u_2 and u_3 as well, using an endomorphism $n \in \mathcal{N}$ that maps u_3 to u_1 and leaves u_1, u_2 fixed. After applying m and n, $\mathcal{K}_3$ "collapses" to the braid $\mathcal{K}_1$.

Definition 4.8 The admissible pair of endomorphisms (m, n) is a *simplifier* for the braid (resp. braid forest) $\mathcal{K} = \langle a, C, D, \pi_A, \pi_B \rangle$ if the following conditions hold:

- $\forall o_1, o_2 \in \mathcal{O}_A(C)$: $m(o_1) = m(o_2)$ implies $n(\pi_A(o_1)) = n(\pi_A(o_2))$,
- $\forall o_1, o_2 \in \mathcal{O}_B(D)$: $n(o_1) = n(o_2)$ implies $m(\pi_B(o_1)) = m(\pi_B(o_2))$.

Given a braid $\mathcal{K} = \langle a, C, D, \pi_A, \pi_B \rangle$ of type A, say, and a simplifier (m, n) for $\mathcal{K}$ one would naively expect to receive a simpler braid just by applying m and n to all components of $\mathcal{K}$. In more detail, we would like to take $a' := m(a)$ as the new root, and to replace C, D, π_A and π_B by

$$
\begin{aligned}
C' &:= \{m(c) \mid c \in C\},\\
D' &:= \{n(d) \mid d \in D\},\\
\pi'_A &:= \{\langle m(o), n(d) \rangle \mid \langle o, d \rangle \in \pi_A\},\\
\pi'_B &:= \{\langle n(o), m(c) \rangle \mid \langle o, c \rangle \in \pi_B\}
\end{aligned}
$$

respectively. Unfortunately, cases exist where $\langle a', C', D', \pi'_A, \pi'_B \rangle$ is (a braid forest but) not a braid. Ignoring technical details that are considered in Section 8 we now give the proper definition of the image of a braid under simplification, ensuring that this image is again a braid.

Definition 4.9 Let (m, n) be a simplifier for the braid $\mathcal{K} = \langle a, C, D, \pi_A, \pi_B \rangle$ of type A, let $\mathcal{K}^{(m,n)} := \langle a', C', D', \pi'_A, \pi'_B \rangle$ be defined as above, and let $\pi' := \pi'_A \cup \pi'_B$. The *image* $\mathcal{K}^{\langle m,n \rangle}$ *of* $\mathcal{K}$ *under* (m, n) is the quintuple $\langle a'', C'', D'', \pi''_A, \pi''_B \rangle$ where

(1) $a'' := a' = m(a)$,
(2) C'' is the set of all elements of C' that are fibered with a'' via π', and D'' is the set of all elements of D' that are fibered with a'' via π',
(3) π''_A is the restriction of π'_A to $\mathcal{O}_A(C'')$, and π''_B is the restriction of π'_B to $\mathcal{O}_B(D'')$.

As a matter of fact, simplification of braids of type B is defined symmetrically. In Section 8 we prove the following result.

Lemma 4.10 *Let (m, n) be a simplifier for the braid $\mathcal{K}$. If $|\mathcal{O}(\mathcal{K})| = |\mathcal{O}(\mathcal{K}^{\langle m,n \rangle})|$, then $\mathcal{K}^{\langle m,n \rangle}$ is a variant of $\mathcal{K}$.*

Definition 4.11 A simplifier (m, n) for the braid $\mathcal{K}$ is called *proper* if

$|\mathcal{O}(\mathcal{K})| > |\mathcal{O}(\mathcal{K}^{\langle m,n\rangle})|$. A braid $\mathcal{K}'$ is called *irreducible* if $\mathcal{K}'$ does not have a proper simplifier.

The following theorem represents the main result concerning braid simplification.

Theorem 4.12 *Let $\mathcal{K}$ be a braid.*

(1) Let $\mathcal{K} = \mathcal{K}_0, \mathcal{K}_1, \ldots, \mathcal{K}_k$ be a sequence of braids such that each braid $\mathcal{K}_{i+1}$ is the braid image of $\mathcal{K}_i$ under a proper simplification, for $i = 0, \ldots, k-1$. Then $k \leq |\mathcal{O}(\mathcal{K})|$.

(2) If $\mathcal{K}_1$ is irreducible and can be simplified to the braid $\mathcal{K}_2$, then $\mathcal{K}_2$ is irreducible and a variant of $\mathcal{K}_1$.

(3) $\mathcal{K}$ can be simplified to an irreducible braid.

(4) If $\mathcal{K}'$ is an irreducible braid that is reached from $\mathcal{K}$ by a sequence of consecutive simplification steps (always taking braid images), then there exists a simplifier (m, n) for $\mathcal{K}$ such that $\mathcal{K}^{\langle m,n\rangle} = \mathcal{K}'$.

(5) If two irreducible braids $\mathcal{K}_1$ and $\mathcal{K}_2$ can be reached from $\mathcal{K}$ by sequences of consecutive simplification steps (always taking braid images), then $\mathcal{K}_1$ and $\mathcal{K}_2$ are variants.

The theorem, which will be proved in Section 8, shows that braid simplification can be considered as the computation of a normal form which is unique modulo variants. In principle we could use the set of all "$\sim$"-equivalence classes of irreducible braids as the underlying domain of the rational amalgamated product. However, for technical convenience we introduce a standard normal form for each braid. The following theorem shows that such a unique normal form can be given. The details of the construction, as well as the proof of the theorem, are given in Section 8 (see Theorems 8.21, 8.22 and Lemma 8.23).

Theorem 4.13 *There exist subsets $\mathcal{O}_A^* \subseteq \mathcal{O}_A$ and $\mathcal{O}_B^* \subseteq \mathcal{O}_B$, and a set of irreducible braids, $A \odot B$, called the* set of braids in standard normal form, *with the following properties:*

(1) Each braid can be simplified to a unique *braid in standard normal form,*

(2) for each braid $\mathcal{K} \in A \odot B$ we have $\mathcal{O}_A(\mathcal{K}) \subseteq \mathcal{O}_A^*$ and $\mathcal{O}_B(\mathcal{K}) \subseteq \mathcal{O}_B^*$,

(3) the elements of $\mathcal{O}_A^*$ and $\mathcal{O}_B^*$ can be enumerated in the form

$$\mathcal{O}_A^* = \{o_\mathcal{K} \mid \mathcal{K} \text{ is a nontrivial braid in standard normal form of type } B\},$$
$$\mathcal{O}_B^* = \{o_\mathcal{K} \mid \mathcal{K} \text{ is a nontrivial braid in standard normal form of type } A\},$$

which means that $\mathcal{K} \mapsto o_\mathcal{K}$ establishes a bijection between the set of all non-trivial braids in standard normal form of type A (B) and $\mathcal{O}_B^*$ ($\mathcal{O}_A^*$).

(4) if $\mathcal{K}$ is a braid in $A \odot B$ with fibering function π, then for each pair $\langle o_{\mathcal{K}'}, e\rangle$ in π always $\mathcal{K}'$ is the unique subbraid of $\mathcal{K}$ with root e.

Note that Property 4 can be interpreted in the sense that every open atom o of $\mathcal{K}$ "stands for" the subbraid with root $e = \pi(o)$, in the sense that the index

$\mathcal{K}'$ of $o = o_{\mathcal{K}'}$ gives the subbraid with root $\pi(o)$. In the sequel, we write $\mathcal{K}_\downarrow$ for the standard normal form of the braid $\mathcal{K}$.

Definition 4.14 The underlying domain of the rational amalgamated product of $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$ is the set $A \odot B$ of all braids in standard normal form.

Let $\mathcal{A}^\Sigma_* := SH^{\mathcal{A}}_{\mathcal{M}}(Z \cup \mathcal{O}^*_A)$, let $\mathcal{B}^\Delta_* := SH^{\mathcal{B}}_{\mathcal{N}}(Z \cup \mathcal{O}^*_B)$. It follows easily that each element of a braid $\mathcal{K} \in A \odot B$ is in $\mathcal{A}^\Sigma_* \cup \mathcal{B}^\Delta_*$. Conversely we have (cf. Lemma 8.24)

Lemma 4.15 *Given $e \in (A_* \cup B_*) \setminus (\mathcal{O}^*_A \cup \mathcal{O}^*_B)$ there exists a unique braid $\mathcal{K} \in A \odot B$ such that e is the root of $\mathcal{K}$.*

4.2 Functions and relations

Given the underlying domain $A \odot B$ of the rational amalgam of $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$, there is now a perfectly natural way to introduce functions and relations that interpret the symbols of the mixed signature $\Sigma \cup \Delta$. Basically, we shall use bijections to "copy" the Σ-structure (Δ-structure) of $\mathcal{A}^\Sigma_*$ ($\mathcal{B}^\Delta_*$) onto $A \odot B$. Consider the functions $root_A : A \odot B \to A_*$ and $root_B : A \odot B \to B_*$:

$$root_A(\mathcal{K}) := \begin{cases} \text{the root of } \mathcal{K} \text{ if } \mathcal{K} \text{ is trivial or has type } A \\ o_{\mathcal{K}} \in \mathcal{O}^*_A \qquad \text{if } \mathcal{K} \text{ is non-trivial and has type } B. \end{cases}$$

$$root_B(\mathcal{K}) := \begin{cases} \text{the root of } \mathcal{K} \text{ if } \mathcal{K} \text{ is trivial or has type } B \\ o_{\mathcal{K}} \in \mathcal{O}^*_B \qquad \text{if } \mathcal{K} \text{ is non-trivial and has type } A. \end{cases}$$

As a direct consequence of Lemma 4.15 we obtain

Lemma 4.16 *The functions $root_A$ and $root_B$ are bijections.*

Here is now the definition of the rational amalgamated product.

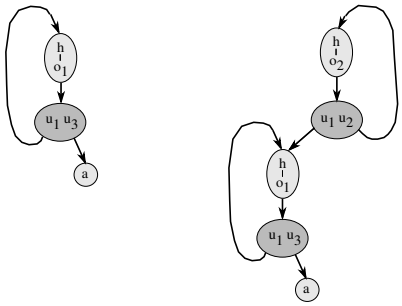
Definition 4.17 The *rational amalgamated product $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$ of $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$* is the following $(\Sigma \cup \Delta)$-structure with carrier $A \odot B$:

(1) Let $f \in \Sigma$ be an n-ary function symbol, let $\mathcal{K}_1, \ldots, \mathcal{K}_n \in A \odot B$. We define $f_{\mathcal{A}\odot\mathcal{B}}(\mathcal{K}_1, \ldots, \mathcal{K}_n) = root_A^{-1}(f_{\mathcal{A}_*}(root_A(\mathcal{K}_1), \ldots, root_A(\mathcal{K}_n)))$.
(2) Let $p \in \Sigma$ be an n-ary predicate symbol, let $\mathcal{K}_1, \ldots, \mathcal{K}_n \in A \odot B$. We define $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta \models p(\mathcal{K}_1, \ldots, \mathcal{K}_n)$ iff $\mathcal{A}^\Sigma_* \models p(root_A(\mathcal{K}_1), \ldots, root_A(\mathcal{K}_n))$.

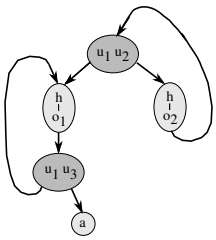
The interpretation of the function symbols $g \in \Delta$ and the predicate symbols $q \in \Delta$ in $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$ is defined symmetrically, using $root_B$.

Example 4.18 Let $\mathcal{A}^\Sigma$ be the freely generated semi-group over the countably infinite set of generators $X = Z \uplus \mathcal{O}_A$, with concatenation as operation, let

$\mathcal{B}^\Delta$ denote the absolutely free algebra for signature $\Delta := \{h, a\}$, where h is unary and a is a constant, with set of generators $Y = Z \uplus \mathcal{O}_B$. The following figure represents two elements $\mathcal{K}_1$ and $\mathcal{K}_2$ of the rational amalgam. We want to define the concatenation of $\mathcal{K}_1$ and $\mathcal{K}_2$.



Both braids are of type B, we have $root_A(\mathcal{K}_1) = u_1$ and $root_A(\mathcal{K}_2) = u_2$. Concatenation of u_1 and u_2 in $\mathcal{A}^\Sigma$ yields the word u_1u_2. Hence the desired result is the unique braid in standard normal form with root u_1u_2, depicted below.



The following theorem shows that the rational amalgamated product — as the free amalgamated product — reflects the structural properties of the component structures in an optimal way. This is one important justification for the rational amalgamation construction.

Theorem 4.19 *As a Σ-structure, $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta, \mathcal{A}^\Sigma$ and $\mathcal{A}^\Sigma_*$ are isomorphic, and $root_A : \mathcal{A}^\Sigma \odot \mathcal{B}^\Delta \to \mathcal{A}^\Sigma_*$ is a Σ-isomorphism. As a Δ-structure, $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta, \mathcal{B}^\Delta$, and $\mathcal{B}^\Delta_*$ are isomorphic, and $root_B : \mathcal{A}^\Sigma \odot \mathcal{B}^\Delta \to \mathcal{B}^\Delta_*$ is a Δ-isomorphism.*

Theorem 4.19 makes clear that rational amalgamation is *not* a construction that can be used, say, to construct the rational tree algebra for a given signature Σ, using the finite tree algebra for Σ as a component. Disregardless of the chosen component $\mathcal{B}^\Delta$, the rational amalgam $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$ is always Σ-isomorphic to $\mathcal{A}^\Sigma$.

4.3 Technical remarks

In this subsection we collect some results that will be needed for proving correctness of the combination algorithms of the following section. The first result is a simple observation about simplification.

Lemma 4.20 *Let (m, n) be a simplifier for the braid (forest) $\mathcal{K}$, let $\mathcal{K}_1$ be the*

unique subbraid of $\mathcal{K}$ with root e, where e is an element of $\mathcal{K}$ of type A (resp. B). Then $\mathcal{K}_1^{\langle m,n\rangle}$ is the unique subbraid of $\mathcal{K}^{(m,n)}$ with root $m(e)$ (resp. $n(e)$).

The proof can be found in Section 8. The next, more interesting result shows that the unique normal form of a braid can always be obtained by a single simplification step.

Lemma 4.21 *Let $\mathcal{K}$ be a braid forest with fibering function π. Let (m,n) denote the admissible pair of endomorphisms that maps each $o \in \mathcal{O}(\mathcal{K})$ to $o_{\mathcal{K}'_\downarrow}$ where $\mathcal{K}'$ is the unique subbraid of $\mathcal{K}$ with root $\pi(o)$. Then (m,n) is a simplifier for $\mathcal{K}$ and each subbraid of $\mathcal{K}^{(m,n)}$ is in standard normal form.*

The proof can be found in Section 8. The simplifier (m,n) will be called the *standard simplifier* for $\mathcal{K}$.

The last result of this subsection follows directly from Lemma 10 of [4].

Lemma 4.22 *Every bijection between $Z \cup \mathcal{O}_A^*$ and $Z \cup \mathcal{O}_A$ extends to a Σ-isomorphism between $\mathcal{A}_*^\Sigma$ and $\mathcal{A}^\Sigma$. Similarly every bijection between $Z \cup \mathcal{O}_B^*$ and $Z \cup \mathcal{O}_B$ extends to a Δ-isomorphism between $\mathcal{B}_*^\Delta$ and $\mathcal{B}^\Delta$.*

5 Combination of Constraint Solvers

Constraint solvers, as considered here, are essentially algorithms that decide solvability of quantifier-free positive formulae in a given solution domain. In this section we show how constraint solvers for two component structures can be combined to a constraint solver for their rational amalgamated product.

5.1 Decidability results for constraints over the rational amalgamated product

In the first subsection we summarize decidability results. We (mostly) disregard disjunction since its integration is a triviality.

Definition 5.1 Let Γ be a signature. A Γ-*constraint* is a conjunction of atomic Γ-formulae.

In order to decide solvability of a "mixed" $(\Sigma \cup \Delta)$-constraint in a rational amalgamated product $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$ we shall decompose it into two pure constraints over the signatures Σ and Δ respectively. These output constraints are equipped with additional restrictions of a particular type.

Definition 5.2 An *A/N (atom/non-atom) declaration* for a constraint γ is a pair (U,W) such that $U \uplus W \subseteq Var(\gamma)$ is a disjoint union. Both U and W may be empty. A solution ν_A of a constraint γ in an SC-structure $(\mathcal{A}^\Sigma, \mathcal{M}, X)$

is called a *solution* of $\langle \gamma, U, W \rangle$ if ν_A assigns distinct atoms to the variables in U, and arbitrary non-atomic elements of A to the variables in W.

In order to avoid some ballast in proofs we shall assume that at least one of the two components is a *non-trivial* SC-structure, which means that it has at least one non-atomic element. We may now formulate our main result concerning combination of constraint solvers in the case of rational amalgamation.

Theorem 5.3 *Let $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$ be two non-collapsing SC-structures over disjoint signatures, let $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$ denote their rational amalgam. Assume that at least one of the two components is a non-trivial SC-structure. Then solvability of $(\Sigma \cup \Delta)$-constraints in $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$ is decidable if solvability of (Σ- resp. Δ-) constraints with A/N declarations is decidable for $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$.*

The proof will be given in Subsection 5.2, it is based on a combination algorithm (Algorithm 1) described in this subsection. There seems to be no general way to characterize solvability of Γ-constraints with A/N declarations in purely logical terms. But for a restricted class of component structures—a class which is of particular interest in the context of rational amalgamation—a logical characterization of the problems that we have to solve in the two component structures can be given.

Definition 5.4 A non-collapsing SC-structure $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ is called *rational* if for every atom $x \in X$ and every element $a \in A$ there exists an endomorphism $m \in \mathcal{M}$ that leaves all atoms $x' \neq x$ fixed such that $m(x) = m(a)$.[8]

The algebra of rational trees over a given signature is always a rational SC-structure. The same holds for feature structures, feature structures with arity, and domains with nested, rational lists (as described in 3.4). For rational SC-structures we obtain the following refinement and reformulation of Theorem 5.3.

Theorem 5.5 *Let $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$ be two non-trivial rational SC-structures over disjoint signatures, let $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$ denote their rational amalgam. Then solvability of $(\Sigma \cup \Delta)$-constraints in $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$ is decidable if the positive universal-existential theory is decidable for both components $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$.*

The proof will be given in Subsection 5.3, it is based on a second combination algorithm (Algorithm 2), which is described in this subsection. Since existential quantification distributes over disjunction, the theorem may be slightly strengthened.

Theorem 5.6 *Let $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$ be two non-trivial rational SC-structures over*

[8] The existence of such an endomorphism is trivial if $x \notin Stab^{\mathcal{A}}_{\mathcal{M}}(a)$. In this case we may always take, e.g., the endomorphism $m = m_{x-a}$ of $\mathcal{M}$ that maps x to a and leaves all other atoms fixed. The situation of interest is the case where $x \in Stab^{\mathcal{A}}_{\mathcal{M}}(a)$ and $x \neq a$.

disjoint signatures, let $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$ *denote their rational amalgam. Then the positive existential theory of* $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$ *is decidable if the positive universal-existential theory is decidable for both components* $\mathcal{A}^\Sigma$ *and* $\mathcal{B}^\Delta$.

It is interesting to contrast this formulation with the corresponding combination result for free amalgamation (Theorem 22 of [4]) which needs *stronger* assumptions on the components: *Let* $\mathcal{A}^\Sigma$ *and* $\mathcal{B}^\Delta$ *be two strong SC-structures over disjoint signatures, let* $\mathcal{A}^\Sigma \otimes \mathcal{B}^\Delta$ *denote their free amalgam. Then the positive existential theory of* $\mathcal{A}^\Sigma \otimes \mathcal{B}^\Delta$ *is decidable if the full positive theory is decidable for both components* $\mathcal{A}^\Sigma$ *and* $\mathcal{B}^\Delta$.

One application of Theorem 5.6 is the following

Corollary 5.7 *Rational amalgamated products* $\mathcal{A}_1^{\Sigma_1} \odot \cdots \odot \mathcal{A}_k^{\Sigma_k}$ *have decidable positive existential theory if the nontrivial components* $\mathcal{A}_i^{\Sigma_i}$ *are rational tree algebras, or nested, rational lists, or feature structures*[9] *, or feature-structures with arity, for* $i = 1, \ldots, k$*, and if the signatures of the components are pairwise disjoint.*

Proof. For all these structures it has been shown that even the full positive theory is decidable, see [4]. □

5.2 The combination algorithm

To prove Theorem 5.3 we shall give an algorithm that reduces a mixed constraint γ in the signature $(\Sigma \cup \Delta)$ non-deterministically to a pair of constraints with A/N declarations over the "pure" signatures Σ and Δ respectively. We shall assume that the input formula γ has the form $\gamma = \gamma_0^\Sigma \wedge \gamma_0^\Delta$ where γ_0^Σ is a conjunction of atomic Σ-formulae, and γ_0^Δ is a conjunction of atomic Δ-formulae. Moreover we assume that γ does not contain any equation between variables. These assumptions do not really restrict the generality of the approach: simple techniques like "variable abstraction", now standard in this area, may be used to transform an arbitrary $(\Sigma \cup \Delta)$-constraint φ into a constraint γ of the form given above, preserving solvability in both directions.

Algorithm 1

The *input* is mixed a constraint $\gamma = \gamma_0^\Sigma \wedge \gamma_0^\Delta$ of the form described above. Let $V_0 = Var(\gamma_0^\Sigma) \cap Var(\gamma_0^\Delta)$ denote the set of *shared* variables of γ. The algorithm has two steps, both are nondeterministic.

Step 1: Variable identification. *Consider all possible partitions of the set*

[9] As in Examples 3.4 we refer to [2], for specificity.

of all shared variables, V_0. Each of these partitions yields one of the new constraints as follows. The variables in each class of the partition are "identified" with each other by choosing an element of the class as representative, and replacing in the input formula all occurrences of variables of the class by this representative.

Step 2: Choose signature labels. *Let $\gamma_1^\Sigma \wedge \gamma_1^\Delta$ denote one of the formulae obtained by Step 1, let V_1 denote the set of representants of shared variables. The set V_1 is partitioned in two subsets U and W in some arbitrary way.*

Let $\sigma = \gamma_1^\Sigma$, let $\delta = \gamma_1^\Delta$. For each of the choices made in Step 1 and 2, the algorithm yields an *output pair* $(\langle \sigma, U, W\rangle, \langle \delta, W, U\rangle)$, each component representing a constraint with A/N declaration. □

Theorem 5.3 is an immediate consequence of the following proposition.

Proposition 5.8 *The input formula γ has a solution in $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$ if and only if there exists an output pair $(\langle \sigma, U, W\rangle, \langle \delta, W, U\rangle)$ of Algorithm 1 such that $\langle \sigma, U, W\rangle$ has a solution in $\mathcal{A}^\Sigma$ and $\langle \delta, W, U\rangle$ has a solution in $\mathcal{B}^\Delta$.*

Basically, the idea behind the combination algorithm may be described as follows. In the first step we guess which variables receive the same value under a (hypothetical) solution. These variables are identified. In the second step we guess which variables are mapped to nontrivial braids of type B (the elements of U) or of type A (the elements of W). Variables that are mapped to trivial braids can be treated arbitrarily. Note that via $root_A$ the former (latter) elements are mapped to open atoms (non-atomic elements) of $\mathcal{A}_*^\Sigma$.

Example 5.9 Let A^Σ and $\mathcal{B}^\Delta$ be defined as in Example 4.18. We consider the constraint with the formulae $h(x_3) \circ y_2 = y_1 \circ h(h(x_1) \circ h(x_2)), x_3 = y_1 \circ a, x_2 = y_1 \circ y_2$. Since the constraint does not have the form required for Algorithm 1 we apply variable abstraction and obtain the equivalent system

$$\begin{aligned} z_1 \circ y_2 &= y_1 \circ z_2 \\ x_3 &= y_1 \circ z_3 \\ x_2 &= y_1 \circ y_2 \\ z_4 &= z_5 \circ z_6 \\ z_1 &= h(x_3) \\ z_2 &= h(z_4) \\ z_5 &= h(x_1) \\ z_6 &= h(x_2) \\ z_3 &= a. \end{aligned}$$

The set of shared variables is $V_0 := \{z_1, z_2, z_3, z_4, z_5, z_6, x_2, x_3\}$.
In the first step of Algorithm 1 we identify z_4 and x_2, z_5 and z_1, as well as z_6 and z_2. In each case we use the latter element as representant. We obtain

(merging two identical equations) the new system

$$\begin{aligned} z_1 \circ y_2 &= y_1 \circ z_2 \\ x_3 &= y_1 \circ z_3 \\ x_2 &= y_1 \circ y_2 \\ x_2 &= z_1 \circ z_2 \\ z_1 &= h(x_3) \\ z_2 &= h(x_2) \\ z_1 &= h(x_1) \\ z_3 &= a \end{aligned}$$

The set of shared representants is $V_1 = \{z_1, z_2, z_3, x_2, x_3\}$.
In the second step of Algorithm 1 we partition this set into the two subsets $U := \{z_1, z_2, z_3\}$ and $W := \{x_2, x_3\}$. The output system for $\mathcal{A}^\Sigma$ contains the equations

$$\begin{aligned} z_1 \circ y_2 &= y_1 \circ z_2 \\ x_3 &= y_1 \circ z_3 \\ x_2 &= y_1 \circ y_2 \\ x_2 &= z_1 \circ z_2 \end{aligned}$$

Here the variables z_1, z_2, z_3 have to be evaluated as distinct atoms, the variables x_2, x_3 as non-atomic elements. We may solve this system with A/N-declaration using the following assignment (u_1, u_2, u_3 denote distinct open atoms of $\mathcal{A}^\Sigma$):

$$z_1, y_1 \mapsto u_1 \mid z_2, y_2 \mapsto u_2 \mid z_3 \mapsto u_3 \mid x_2 \mapsto u_1 \circ u_2 \mid x_3 \mapsto u_1 \circ u_3.$$

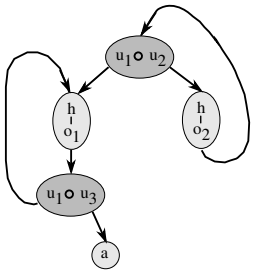
The output system for $\mathcal{B}^\Delta$ contains the equations

$$\begin{aligned} z_1 &= h(x_3) \\ z_2 &= h(x_2) \\ z_1 &= h(x_1) \\ z_3 &= a \end{aligned}$$

In this case the variables x_2 and x_3 have to be evaluated as distinct atoms, the variables z_1, z_2, z_3 as non-atomic elements. The following is a solution:

$$x_1, x_3 \mapsto o_1 \mid x_2 \mapsto o_2 \mid z_1 \mapsto h_1(o_1) \mid z_2 \mapsto h(o_2) \mid z_3 \mapsto a.$$

Since both output systems are solvable, the input constraint has a solution in $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$. The following figure depicts such a solution. To identify the value, say, of x_3 (which is mapped to $u_1 \circ u_3$ and o_1 respectively under the above component solutions) consider the subbraid with "name" o_1, i.e, the subbraid with root $u_1 \circ u_3$. The values of the other variables can be read off in the same way.



Correctness of Algorithm 1

In order to prove Proposition 5.8 we shall assume that the two components $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$ are non-collapsing SC-structures of the form $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ and $(\mathcal{B}^\Delta, \mathcal{N}, Y)$ respectively. First we show soundness.

Lemma 5.10 *If, for some output pair* $(\langle \sigma, U, W\rangle, \langle \delta, W, U\rangle)$ *of Algorithm 1,* $\langle \sigma, U, W\rangle$ *has a solution in* $\mathcal{A}^\Sigma$ *and* $\langle \delta, W, U\rangle$ *has a solution in* $\mathcal{B}^\Delta$*, then the input constraint* γ *is solvable in* $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$*.*

Proof. The output formulae σ and δ may be written in the form $\gamma_1^\Sigma(\vec{u}, \vec{w}, \vec{v}_\Sigma)$ and $\gamma_1^\Delta(\vec{u}, \vec{w}, \vec{v}_\Delta)$, where $\vec{u} = u_1, \ldots, u_m$ denotes the sequence of all elements of U, where $\vec{w} = w_1, \ldots, w_n$ denotes the sequence of all elements of W, and where $\vec{v}_\Sigma$ (resp. $\vec{v}_\Delta$) stands for the non-shared variables occurring in γ_1^Σ and γ_1^Δ respectively. The proof has now three steps. In the first step, the given solutions of the output constraints are used to construct similar solutions of a more specific form. In the second step, these latter solutions are used to define suitable braids. In the third step we apply standard normalization to these braids. This will yield a solution of the input constraint.

1. By assumption, there exists a solution μ_A of γ_1^Σ in $\mathcal{A}^\Sigma$ such that the elements $\mu_A(u_1), \ldots, \mu_A(u_m)$ are distinct atoms of $\mathcal{A}^\Sigma$, and the elements $\mu_A(w_1), \ldots, \mu_A(w_n)$ are non-atomic elements of $\mathcal{A}^\Sigma$. If some of the atoms $\mu_A(u_1), \ldots, \mu_A(u_m)$ are bottom atoms, then we apply an automorphism $m_1 \in \mathcal{M}$ such that the elements in $\{m_1(\mu_A(u_1)), \ldots, m_1(\mu_A(u_m))\}$ are distinct open atoms. In the other case, let $m_1 := Id$. If the stabilizers of the elements $m_1(\mu_A(w_1)), \ldots, m_1(\mu_A(w_n))$ contain open atoms $o_1, \ldots, o_k$ that do not belong to $\{m_1(\mu_A(u_1)), \ldots, m_1(\mu_A(u_m))\}$, then we apply an endomorphism m_2 that maps the atoms $o_1, \ldots, o_k$ to some bottom atom z and leaves the atoms $\{m_1(\mu_A(u_1)), \ldots, m_1(\mu_A(u_m))\}$ fixed. In the other case, let $m_2 := Id$. Since γ_1^Σ is a positive formula, $\nu_A := \mu_A \circ m_1 \circ m_2$ is a solution of γ_1^Σ, by Lemma 3.5. We have

(1) the elements $x_1 := \nu_A(u_1), \ldots, x_m := \nu_A(u_m)$ are distinct *open* atoms,
(2) the elements $a_1 := \nu_A(w_1), \ldots, a_n := \nu_A(w_n)$ are non-atomic,
(3) the open atoms occurring in the stabilizers of the elements $a_1, \ldots, a_n$ are in $\{x_1, \ldots, x_m\}$, and

(4) $\mathcal{A}^\Sigma \models \exists \vec{v}_\Sigma\ \gamma_1^\Sigma(\vec{u}/\vec{x}, \vec{w}/\vec{a})$.

(2) follows from the fact $\mathcal{A}^\Sigma$ is non-collapsing, (3) follows from Lemma 3.10, and (4) follows from the fact that ν_A solves γ_1^Σ. Symmetrically we can show that there exists a solution ν_B of γ_1^Δ in $\mathcal{B}^\Delta$ such that

(5) the elements $y_1 := \nu_B(w_1), \ldots, y_n := \nu_B(w_n)$ are distinct *open* atoms,
(6) the elements $b_1 := \nu_B(u_1), \ldots, b_m := \nu_B(u_m)$ are non-atomic,
(7) the open atoms occurring in the stabilizers of the elements $b_1, \ldots, b_m$ are in $\{y_1, \ldots, y_n\}$, and
(8) $\mathcal{B}^\Delta \models \exists \vec{v}_\Delta\ \gamma_1^\Delta(\vec{u}/\vec{b}, \vec{w}/\vec{y})$.

2. Let $\pi_A := \{\langle x_i, b_i\rangle \mid i = 1, \ldots, m\}$, let $\pi_B := \{\langle y_i, a_i\rangle \mid i = 1, \ldots, n\}$. Properties (1)–(3) and (5)–(7) show that for each $e \in \vec{a}$ ($e \in \vec{b}$), the tuple $\mathcal{K}_e := \langle e, \{a_1, \ldots, a_n\}, \{b_1, \ldots, b_m\}, \pi_A, \pi_B\rangle$ is a braid forest of type A (B). These braid forests just differ in the choice of the root.

3. Fix some $e \in \vec{a} \cup \vec{b}$. Let (m_3, n_3) be the standard normalizer for $\mathcal{K}_e$. By Lemma 3.5, (4), and (8),

$$\mathcal{A}^\Sigma \models \exists \vec{v}_\Sigma\ \gamma_1^\Sigma(\vec{u}/m_3(\vec{x}), \vec{w}/m_3(\vec{a})),$$
$$\mathcal{B}^\Delta \models \exists \vec{v}_\Delta\ \gamma_1^\Delta(\vec{u}/n_3(\vec{b}), \vec{w}/n_3(\vec{y})).$$

It follows easily from Lemma 4.22 that

$$\mathcal{A}^\Sigma_* \models \exists \vec{v}_\Sigma\ \gamma_1^\Sigma(\vec{u}/m_3(\vec{x}), \vec{w}/m_3(\vec{a})),$$
$$\mathcal{B}^\Delta_* \models \exists \vec{v}_\Delta\ \gamma_1^\Delta(\vec{u}/n_3(\vec{b}), \vec{w}/n_3(\vec{y})).$$

Now Theorem 4.19 shows that

$$\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta \models \exists \vec{v}_\Sigma\ \gamma_1^\Sigma(\vec{u}/root_A^{-1}(m_3(\vec{x})), \vec{w}/root_A^{-1}(m_3(\vec{a}))),$$
$$\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta \models \exists \vec{v}_\Delta\ \gamma_1^\Delta(\vec{u}/root_B^{-1}(n_3(\vec{b})), \vec{w}/root_B^{-1}(n_3(\vec{y}))).$$

We shall now show that for each $u_i \in \vec{u}$ the two images $root_A^{-1}(m_3(x_i))$ and $root_B^{-1}(n_3(b_i))$ of u_i mentioned in the previous expressions coincide, and similarly for the two images of the elements $w_j \in \vec{w}$.

Consider an element x_i of $\vec{x}$. Recall that $b_i = \pi_A(x_i)$. Let $\mathcal{K}'$ denote the unique subbraid of $\mathcal{K}_e$ with root b_i. The definition of standard normalization shows that $m_3(x_i) = o_{\mathcal{K}'_\downarrow}$. Let $\mathcal{K}_i$ be the subbraid of $\mathcal{K}_e^{(m_3,n_3)}$ with root $n_3(b_i)$. By Lemmas 4.20 and 4.21, $\mathcal{K}_i = \mathcal{K}'_\downarrow$, hence we have $m_3(x_i) = o_{\mathcal{K}_i}$. The braid $\mathcal{K}_i$ is non-trivial and of type B, and it is the unique braid in standard normal form with root $n_3(b_i)$ (Lemma 4.21, Lemma 4.15). Hence $root_A^{-1}(m_3(x_i)) = \mathcal{K}_i$, by definition of $root_A$. The element $n_3(b_i)$ is a non-atomic element of B. Hence

$root_B^{-1}(n_3(b_i)) = \mathcal{K}_i$ is the unique braid in standard normal form with root $n_3(b_i)$. Thus we have seen that $root_A^{-1}(m_3(\vec{x})) = root_B^{-1}(n_3(\vec{b}))$.

Similarly it follows that $root_B^{-1}(n_3(\vec{y})) = root_A^{-1}(m_3(\vec{a}))$. This shows that the formula $\gamma_1^\Sigma \wedge \gamma_1^\Delta$ obtained after Step 1 has a solution in $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$. Obviously this implies that the input constraint γ has a solution in $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$. □

Next, we show completeness of the Algorithm 1.

Proposition 5.11 *If the input constraint γ has a solution in $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$, then there exists an output pair $(\langle \sigma, U, W\rangle, \langle \delta, W, U\rangle)$ of Algorithm 1 such that $\langle \sigma, U, W\rangle$ has a solution in $\mathcal{A}^\Sigma$ and $\langle \delta, W, U\rangle$ has a solution in $\mathcal{B}^\Delta$.*

Proof. Assume that γ has a solution $\mu_{A\odot B}$ in $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$.

In Step 1 of Algorithm 1 we identify two shared variables v and v' of V_0 if, and only if, $\mu_{A\odot B}(v) = \mu_{A\odot B}(v')$. With this choice, $\mu_{A\odot B}$ is a solution of the formula $\gamma_1^\Sigma \wedge \gamma_1^\Delta$ that is reached after Step 1, and $\mu_{A\odot B}$ assigns distinct values in $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$ to all variables of V_1.

By Theorem 4.19, $\mu_{A\odot B} \circ root_A$ (resp. $\mu_{A\odot B} \circ root_B$) is a solution of $\sigma = \gamma_1^\Sigma$ in $\mathcal{A}_*^\Sigma$ (resp. of $\delta = \gamma_1^\Delta$ in $\mathcal{B}_*^\Delta$) that does not identify two variables of V_1.

By assumption, one of the two component structures, $\mathcal{A}^\Sigma$, say, is non-trivial. In Step 2, we choose as U the set of all variables u of V_1 such that $\mu_{A\odot B}(u)$ is a non-trivial braid of type B. Consequently, W contains all variables w of V_1 such that $\mu_{A\odot B}(w)$ is a trivial braid or a non-trivial braid of type A. The definition of $root_A$ implies that $\mu_{A\odot B} \circ root_A(u)$ is an open atom of $\mathcal{A}_*^\Sigma$, for all $u \in U$, and $\mu_{A\odot B} \circ root_A(w)$ is a non-atomic element or a bottom atom of $\mathcal{A}_*^\Sigma$, for all $w \in W$. Let $m_1 \in \mathcal{M}$ be an endomorphism that maps all the bottom atoms of the set $\{\mu_{A\odot B} \circ root_A(w) \mid w \in W\}$ to a non-atomic element of A and leaves all other atoms fixed. Since $\mathcal{A}^\Sigma$ is non-collapsing, all elements of the set $\{\mu_{A\odot B} \circ root_A \circ m_1(w) \mid w \in W\}$ are non-atomic. Since σ is a positive formula, Lemma 3.5 implies that $\nu_A := \mu_{A\odot B} \circ root_A \circ m_1$ is a solution of $\langle \sigma, U, W\rangle$ in $\mathcal{A}_*^\Sigma$.

On the other hand the definition of $root_B$ implies that $\mu_{A\odot B} \circ root_B(w)$ is an atom of $\mathcal{B}_*^\Delta$, for all $w \in W$, and $\mu_{A\odot B} \circ root_B(u)$ is a non-atomic element of $\mathcal{B}_*^\Delta$, for all $u \in U$. This shows that $\langle \delta, W, U\rangle$ has a solution in $\mathcal{B}_*^\Delta$.

But then, by Lemma 4.22, $\langle \sigma, U, W\rangle$ has a solution in $\mathcal{A}^\Sigma$ and $\langle \delta, W, U\rangle$ has a solution in $\mathcal{B}^\Delta$. □

In order to proof Theorem 5.5 we shall use the following variant of Algorithm 1, which we call

Algorithm 2

The input constraint γ, and Steps 1 and 2, remain as above. The output of Algorithm 2 consists of the two positive universal-existential sentences

$$\sigma = \forall\vec{u}\exists\vec{w}\exists\vec{v}_{1,\Sigma}\ \gamma_1^\Sigma$$

and

$$\delta = \forall\vec{w}\exists\vec{u}\exists\vec{v}_{1,\Delta}\ \gamma_1^\Delta$$

where $\vec{u}$ ($\vec{w}$) represent the variables in U (resp. W), $\vec{v}_{1,\Sigma}$ represents the non-shared variables in γ_1^Σ, and $\vec{v}_{1,\Delta}$ represents the non-shared variables in γ_1^Δ.

Proposition 5.12 *The input formula γ has a solution in $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$ if and only if there exists an output pair (σ, δ) of Algorithm 2 such that $\mathcal{A}^\Sigma \models \sigma$ and $\mathcal{B}^\Delta \models \delta$.*

Theorem 5.5 is an immediate consequence. In order to prove Proposition 5.12 we shall first show that Algorithm 2 is sound. As above we shall assume that the two components $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$ have the form $(\mathcal{A}^\Sigma, \mathcal{M}, X)$ and $(\mathcal{B}^\Delta, \mathcal{N}, Y)$ respectively.

Lemma 5.13 *If, for some output pair (σ, δ) of Algorithm 2, $\mathcal{A}^\Sigma \models \sigma$ and $\mathcal{B}^\Delta \models \delta$, then γ is solvable in $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$.*

Proof. Assume that $\mathcal{A}^\Sigma \models \forall\vec{u}\exists\vec{w}\exists\vec{v}_{1,\Sigma}\ \gamma_1^\Sigma$ and $\mathcal{B}^\Delta \models \forall\vec{w}\exists\vec{u}\exists\vec{v}_{1,\Delta}\ \gamma_1^\Delta$. Let $\vec{u} = u_1, \ldots, u_m$, let $\vec{w} = w_1, \ldots, w_n$. For each variable u_i we select a distinct atom $x_i \in X$ of A ($1 \leq i \leq m$), and for each variable w_j we select a distinct atom $y_j \in Y$ of B ($1 \leq j \leq n$). Then there are elements $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in B$ such that

$$\mathcal{A}^\Sigma \models \exists\vec{v}_{1,\Sigma}\ \gamma_1^\Sigma(\vec{u}/\vec{x}, \vec{w}/\vec{a})$$
$$\mathcal{B}^\Delta \models \exists\vec{v}_{1,\Delta}\ \gamma_1^\Delta(\vec{u}/\vec{b}, \vec{w}/\vec{y}).$$

We distinguish two cases.

First case: $x_i \neq a_j$ and $b_i \neq y_j$, for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Since $\mathcal{A}^\Sigma$ is non-trivial, we may choose an endomorphism $m_1 \in \mathcal{M}$ that maps all atoms in the set $\{a_1, \ldots, a_n\}$ to a non-atomic element $a \in A$ and fixes all other atoms.

In particular, m_1 leaves the atoms $x_1, \ldots, x_m$ fixed, by assumption. Since $\mathcal{A}^\Sigma$ is non-collapsing, all elements in the set $\{m_1(a_1), \ldots, m_1(a_n)\}$ are non-atomic. Since γ_1^Σ is a positive formula we have

$$\mathcal{A}^\Sigma \models \exists \vec{v}_{1,\Sigma}\ \gamma_1^\Sigma(\vec{u}/\vec{x}, \vec{w}/\vec{m}_1(a)),$$

by Lemma 3.5. It follows that the Σ-constraint with A/N declaration, (γ_1^Σ, U, W), has a solution in $\mathcal{A}^\Sigma$.

Symmetrically we may choose an endomorphism $n_1 \in \mathcal{N}$ such that all elements in $\{n_1(b_1), \ldots, n_1(b_m)\}$ are non-atomic and

$$\mathcal{B}^\Delta \models \exists \vec{v}_{1,\Delta}\ \gamma_1^\Delta(\vec{u}/n_1(\vec{b}), \vec{w}/y).$$

It follows that the Δ-constraint with A/N declaration, (γ_1^Δ, W, U), has a solution in $\mathcal{B}^\Delta$. Now Lemma 5.10 shows that the input formula γ has a solution in $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$.

Second case: Without loss of generality, $x_i = a_j$, for some $1 \leq i \leq m$ and $1 \leq j \leq n$. We consider the new formula $\gamma'_{1,\Sigma}$ ($\gamma'_{1,\Delta}$) that is obtained by replacing all occurrences of w_j in γ_1^Σ (resp. γ_1^Δ) by u_i. Consider the pair with the formulae $\sigma' = \forall\vec{u}\exists\vec{w}'\exists\vec{v}_{1,\Sigma}\ \gamma'_{1,\Sigma}$ and $\delta' = \forall\vec{w}'\exists\vec{u}\exists\vec{v}_{1,\Delta}\ \gamma'_{1,\Delta}$, where the sequence $\vec{w}'$ is obtained from $\vec{w}$ by removing w_j. Obviously, (σ', δ') is again an output pair Algorithm 2. We claim that $\mathcal{A}^\Sigma \models \sigma'$ and $\mathcal{B}^\Delta \models \delta'$.

We have

$$\mathcal{A}^\Sigma \models \exists \vec{v}_{1,\Sigma}\ \gamma'_{1,\Sigma}(\vec{u}/\vec{x}, \vec{w}'/\vec{a}'),$$

where $\vec{a}'$ denotes the sequence $a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n$. Since X is an $\mathcal{M}$-atom set, for each sequence $\vec{c} = c_1, \ldots, c_m$ of elements of A there exists an endomorphism $m_2 \in \mathcal{M}$ such that $m_2(x_i) = c_i$, for $1 \leq i \leq m$. Now Lemma 3.5 shows that $\mathcal{A}^\Sigma \models \sigma'$.

Since $(\mathcal{B}^\Delta, \mathcal{N}, Y)$ is rational, there exists an endomorphism $n_2 \in \mathcal{N}$ that leaves all atoms but y_j fixed such that $n_2(y_j) = n_2(b_i)$. By Lemma 3.5,

$$\mathcal{B}^\Delta \models \exists \vec{v}_{1,\Delta}\ \gamma'_{1,\Delta}(\vec{u}/n_2(\vec{b}), \vec{w}'/\vec{y}'),$$

where the sequence $\vec{y}'$ is obtained from $\vec{y}$ by removing y_j. Since the elements in the sequence $\vec{y}'$ are distinct atoms it follows as above that $\mathcal{B}^\Delta \models \delta'$.

In this second case we have seen that we can construct a new output pair (σ', δ') of Algorithm 2 such that $\mathcal{A}^\Sigma \models \sigma'$ and $\mathcal{B}^\Delta \models \delta'$. Moreover, the number of variables in (σ', δ') is strictly smaller than the number of variables in (σ, δ). We may now use the same subcase analysis as above, replacing (σ, δ) by (σ', δ'), and iterate this contraction of formulae, if necessary. After a finite number of steps we reach an output pair that satisfies all the assumptions that we made for (σ, δ) in the first subcase. As we have seen, this shows that the input formula γ has a solution in $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$. □

As the last step, we show completeness of Algorithm 2.

Lemma 5.14 *If the input constraint γ has a solution in $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$, then there exists an output pair (σ, δ) of Algorithm 2 such that $\mathcal{A}^\Sigma \models \sigma$ and $\mathcal{B}^\Delta \models \delta$.*

Proof. Lemma 5.11 shows that Algorithm 1 has an output pair $(\langle \gamma_1^\Sigma, U, W\rangle, \langle \gamma_1^\Delta, W, U\rangle)$ such that $\langle \gamma_1^\Sigma, U, W\rangle$ has a solution in $\mathcal{A}^\Sigma$ and $\langle \gamma_1^\Delta, W, U\rangle$ has a solution in $\mathcal{B}^\Delta$. In $\mathcal{A}^\Sigma$, variables of U are interpreted as distinct atoms in X under the given solution. Lemma 3.6 shows that $\mathcal{A}^\Sigma \models \forall\vec{u}\exists\vec{w}\exists\vec{v}_{1,\Sigma}\ \gamma_1^\Sigma$. In $\mathcal{B}^\Delta$, variables of W are interpreted as distinct atoms in Y under the given solution. By Lemma 3.6, $\mathcal{B}^\Delta \models \forall\vec{w}\exists\vec{u}\exists\vec{v}_{1,\Delta}\ \gamma_1^\Delta$. This shows that the sentences $\sigma := \forall\vec{u}\exists\vec{w}\exists\vec{v}_{1,\Sigma}\ \gamma_1^\Sigma$ and $\delta := \forall\vec{w}\exists\vec{u}\exists\vec{v}_{1,\Delta}\ \gamma_1^\Delta$ of the corresponding output pair (σ, δ) of Algorithm 2 are valid in $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$ respectively. □

6 Free Amalgamation and Rational Amalgamation

In this subsection we define the notion of an *acyclic* braid and show that the set of all acyclic braids in standard normal form is a substructure of the rational amalgamated product.

Definition 6.1 A braid forest $\mathcal{K} = \langle a, C, D, \pi_A, \pi_B\rangle$ is called *acyclic* if there is no sequence $e_1, e_2 \ldots, e_n$ of elements in $C \cup D$, of length $n \geq 2$, such that $e_1 = e_n$ and every element e_i is directly fibered via $\pi = \pi_A \cup \pi_B$ with e_{i+1}, for $i = 1, \ldots, n-1$. If $\mathcal{K}$ is acyclic, the *depth* of $\mathcal{K}$ is the largest number n such that there is a sequence $e_1, \ldots, e_n$ of elements of $\mathcal{K}$ where each element e_i is directly fibered with e_{i+1} via π, for $i = 1, \ldots, n-1$.

Lemma 6.2 *The set of all acyclic braids of $A \odot B$ forms a substructure of $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$.*

Proof. Let $f \in \Sigma$ be an n-ary function symbol, let $\mathcal{K}_1, \ldots, \mathcal{K}_n$ be acyclic elements of $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$. We have to show that the braid

$$f_{\mathcal{A}\odot\mathcal{B}}(\mathcal{K}_1, \ldots, \mathcal{K}_n) = root_A^{-1}(f_{\mathcal{A}_*}(root_A(\mathcal{K}_1), \ldots, root_A(\mathcal{K}_n)))$$

is acyclic. Each element o of $\mathcal{O}_A(\{root_A(\mathcal{K}_1), \ldots, root_A(\mathcal{K}_n)\}$ has the form $o = o_\mathcal{K}$ where the braid $\mathcal{K}$ in standard normal form is acyclic. By Lemma 3.9,

$$\mathcal{O}_A(f_{\mathcal{A}_*}(root_A(\mathcal{K}_1), \ldots, root_A(\mathcal{K}_n)) \subseteq \mathcal{O}_A(\{root_A(\mathcal{K}_1), \ldots, root_A(\mathcal{K}_n)\}$$

and the open atoms in $a^* := f_{\mathcal{A}_*}(root_A(\mathcal{K}_1), \ldots, root_A(\mathcal{K}_n))$ represent acyclic subbraids. If $a^* \in \mathcal{A}_* \setminus \mathcal{O}_A^*$, then $\mathcal{K} := root_A^{-1}(a^*)$ is the unique braid in standard normal form with root a^*. Since all open atoms of the root of $\mathcal{K}$ represent acyclic subbraids, $\mathcal{K}$ itself is acyclic. In the other case, if $a^* = o_\mathcal{K} \in \mathcal{O}_A(\{root_A(\mathcal{K}_1), \ldots, root_A(\mathcal{K}_n)\}$ is an atom, then $\mathcal{K} = root_A^{-1}(a^*)$ is acyclic.

We have seen that the set of all acyclic braids represents a Σ-substructure of $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$. Symmetrically it follows that this set represents a Δ-substructure of $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$. □

The relevance of Lemma 6.2 relies on the following theorem, which yields further evidence for the naturalness of rational amalgamation.

Theorem 6.3 *Let $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$ be two strong and non-collapsing*[10] *SC-structures over disjoint signatures. The free amalgamated product of $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$ (as introduced in [4]) is isomorphic to the substructure of $\mathcal{A}^\Sigma \odot \mathcal{B}^\Delta$ consisting of all acyclic braids.*

The proof of this theorem needs additional background from free amalgamation that cannot be given here. We refer to the thesis [12] of the second co-author. For readers that are familiar with the definition of the free amalgamated product let us point out that induction on the depth of an acyclic braid may be used to construct the factorising homomorphisms that characterize the free amalgamated product up to isomorphism.

7 Rational Amalgamation of Rational Trees – A Case Study

When we formalized the mixed elements of the rational amalgam using the concept of a braid our aim was to obtain a construction that is as general as possible. Still, for some wellknown component structures our intuition says that a simpler description of combined elements should be possible. For example, if the pure elements of the components can be described in the form of ordered trees, then we expect that braids essentially behave like rational trees. This intuition was even emphasized in the discussion of our motivating Example 4.1. We now show that both descriptions are in fact equivalent, in a precise sense to be explained. For simplicity and specificity we concentrate on the case where the component structures are rational tree algebras. The main result is captured by the following theorem.

Theorem 7.1 *The rational amalgamated product of two algebras of rational trees over disjoint signatures is isomorphic to the algebra of rational trees over the combined signature.*

Before we give the proof, let us fix notational details. We consider two non-ground algebras of rational trees of the form $\mathcal{A}^\Sigma = R(\Sigma, X)$ and $\mathcal{B}^\Delta = R(\Delta, Y)$ respectively, where the signatures Σ and Δ are disjoint.[11]. As in Section 4

[10] With the actual methods, the free amalgamated product can only be built for *strong* SC-structures over disjoint signatures. Hence we have to assume that the two components are strong and non-collapsing.

[11] See [8] for the definition of the algebra of rational trees over a given signature.

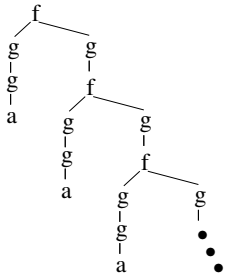
we assume that sets of atoms (i.e., variables) for $\mathcal{A}^\Sigma$ and $\mathcal{B}^\Delta$ have the form $X = Z \uplus \mathcal{O}_B$ and $Y = Z \uplus \mathcal{O}_B$ respectively, where $Z = X \cap Y$ is an infinite set of bottom atoms. Some characteristics of the present situation are the following

- When treating the two components as SC-structures, the relevant monoids of endomorphisms are $\mathcal{M} := End^\Sigma_\mathcal{A}$ and $\mathcal{N} := End^\Sigma_\mathcal{B}$ respectively.
- For a rational tree t, the stabilizer of t is just the set of all atoms (i.e., variables) that label a leaf of t.
- If m is an admissible endomorphism of $R(\Sigma, X)$, then $Stab^\mathcal{A}_\mathcal{M}(m(t)) = \{m(x) \mid x \in Stab^\mathcal{A}_\mathcal{M}(t)\}$. From this equality, and from the corresponding equality for $R(\Delta, Y)$ it follows easily that $\mathcal{K}^{(m,n)} = \mathcal{K}^{\langle m,n\rangle}$ (compare Def. 4.8), for all braids $\mathcal{K}$ over $R(\Sigma, X)$ and $R(\Delta, Y)$ and all admissible pairs of endomorphisms (m, n).

Consider a braid $\mathcal{K} = \langle a, C, D, \pi_A, \pi_B\rangle$ in $R(\Sigma, X) \odot R(\Delta, Y)$. The fibering function $\pi := \pi_A \cup \pi_B$ can be considered as a mapping that replaces each hole (i.e., open atom) o of a context in C or D by a subcontext $\pi(o)$. The process where we apply π to the root a of $\mathcal{K}$ in an iterative way, obtaining a sequence $a_0 = a, a_1 = \pi(a), \ldots, a_{n+1} = \pi(a_n), \ldots$ is called *unfolding* of $\mathcal{K}$. Since, by Definition 4.4, π replaces open atoms by non-atomic elements, the above sequence converges (w.r.t. the usual metrics on infinite trees) to a unique limit tree $t_\mathcal{K}$ that does not have any open atom. We define the mapping

$$unfold : R(\Sigma, X) \odot R(\Delta, Y) \to R(\Sigma \cup \Delta, Z) : \mathcal{K} \mapsto t_\mathcal{K}.$$

Example 7.2 The following figure gives the rational tree that is obtained by unfolding the braid from Example 4.1:



Now Theorem 7.1 is an immediate consequence of

Proposition 7.3 *The mapping unfold is an isomorphism between* $R(\Sigma, X) \odot R(\Delta, Y)$ *and the algebra of rational trees over the joint signature,* $R(\Sigma \cup \Delta, Z)$.

Proof. The proof consists of four parts.

(1) We define a mapping *fold* that assigns a braid in $R(\Sigma, X) \odot R(\Delta, Y)$ to each rational tree in $R(\Sigma \cup \Delta, Z)$.
(2) We show that $t = unfold(fold(t))$, for each $t \in R(\Sigma \cup \Delta, Z)$. This shows that *fold* is injective and *unfold* is surjective.
(3) We show that $\mathcal{K} = fold(unfold(\mathcal{K}))$, for each braid $\mathcal{K}$ in $R(\Sigma, X) \odot$

$R(\Delta, Y)$. Hence we know that *fold* and *unfold* are inverse bijections.

(4) We show that *unfold* and *fold* are $(\Sigma \cup \Delta)$-homomorphisms.

Part 1. A tree $t \in R(\Sigma \cup \Delta, Z)$ is called a Σ-tree (Δ-tree) if the topmost function symbol of t belongs to Σ (resp. Δ). Let $t \in R(\Sigma \cup \Delta, Z)$. Each node η of t defines a unique occurrence of a subtree t_η of t in the obvious way. This occurrence is called *relevant* if either η is the root of t (hence $t_\eta = t$) or if the topmost function symbol of t_η and the label of the father node of η belong to distinct signatures. A subtree t' of t is called relevant if t' has at least one relevant occurrence in t.

When translating rational trees to braids, the idea is basically to replace relevant occurrences of subtrees by open atoms that represent these subtrees. To this end, let op^A (resp. op^B) be a 1-1 mapping that assigns to each Δ-tree (resp. Σ-tree) $t \in R(\Sigma \cup \Delta, Z)$ an open atom $op^A(t) \in \mathcal{O}_A$ (resp. $op^B(t) \in \mathcal{O}_B$). These mappings can be used to define "purifying" 1-1 functions

$$pur^\Sigma : R(\Sigma \cup \Delta, Z) \to R(\Sigma, X)$$
$$pur^\Delta : R(\Sigma \cup \Delta, Z) \to R(\Delta, Y)$$

as follows. Both pur^Σ and pur^Δ fix all atoms $z \in Z$. Moreover, $pur^\Sigma(t) := op^A(t)$ for each Δ-tree t, and conversely $pur^\Delta(t) := op^B(t)$ for each Σ-tree t. If t has the form $t = f(t_1, \ldots, t_n)$ with $f \in \Sigma$, then $pur^\Sigma(t) = f(pur^\Sigma(t_1), \ldots, pur^\Sigma(t_n))$. Symmetrically $pur^\Delta(t)$ behaves homomorphic with respect to function symbols from Δ. We define a braid representation $fold^*(t)$ of a rational tree $t \in R(\Sigma \cup \Delta, Z)$ as follows.

(1) If t is an atom $z \in Z$, then $fold^*(t)$ is the trivial braid with root z.

(2) If t is a Σ-tree, then $fold^*(t)$ is the braid $\langle a, C, D, \pi_A, \pi_B \rangle$ with the following components:
   (a) $a := pur^\Sigma(t)$,
   (b) $C = \{pur^\Sigma(t_1), \ldots, pur^\Sigma(t_k)\}$, where $t_1, \ldots, t_k$ are the relevant Σ-subtrees of t,
   (c) $D = \{pur^\Delta(t'_1), \ldots, pur^\Delta(t'_l)\}$, where $t'_1, \ldots, t'_l$ are the relevant Δ-subtrees of t,
   (d) $\pi_A := \{\langle op^A(t'_i), pur^\Delta(t'_i) \rangle \mid i = 1, \ldots, l\}$,
   (e) $\pi_B := \{\langle op^B(t_i), pur^\Sigma(t_i) \rangle \mid i = 1, \ldots, k\}$.

(3) If t is a Δ-tree, then $fold^*(t)$ is defined symmetrically, using op^B and pur^Δ instead of op^A and pur^Σ.

We may now define the mapping *fold*.

$$fold : R(\Sigma \cup \Delta, Z) \to R(\Sigma, X) \odot R(\Delta, Y)$$

assigns to each rational tree t the unique element of $R(\Sigma, X) \odot R(\Delta, Y)$ that represents the standard normal form of the braid $fold^*(t)$.

Part 2. Let $t \in R(\Sigma \cup \Delta, Z)$. Clearly we obtain t back again by unfolding $\mathit{fold}^*(t)$. Moreover, it is easy to see that the result of the unfolding process is not influenced by simplification. Hence $t = \mathit{unfold}(\mathit{fold}(t))$.

Part 3. Let $\mathcal{K} \in R(\Sigma, X) \odot R(\Delta, Y)$ be an arbitrary braid in standard normal form. In order to prove that $\mathcal{K} = \mathit{fold}(\mathit{unfold}(\mathcal{K}))$ we show that the step from $\mathcal{K}$ to $\mathit{unfold} \circ \mathit{fold}^*(\mathcal{K})$ can be described as a simplification of $\mathcal{K}$. Since $\mathit{unfold} \circ \mathit{fold}(\mathcal{K})$ is obtained from $\mathit{unfold} \circ \mathit{fold}^*(\mathcal{K})$ by an additional (standard) simplification step, and since both $\mathcal{K}$ and $\mathit{unfold} \circ \mathit{fold}(\mathcal{K})$ are in standard normal form, this implies that $\mathcal{K} = \mathit{unfold} \circ \mathit{fold}(\mathcal{K})$, by Theorem 4.13.

Let $\mathcal{K}$ as above, and let π be the fibering function of $\mathcal{K}$. In order to show that $\mathit{unfold} \circ \mathit{fold}$ is a simplification, we define an admissible pair of endomorphisms (m_1, n_1) as follows. Let $o' \in \mathcal{O}_A(\mathcal{K})$. Then $m_1(o') := \mathit{op}^A(\mathit{unfold}(\mathcal{K}'))$, where $\mathcal{K}'$ denotes the unique subbraid of $\mathcal{K}$ with root $d := \pi(o')$. Symmetrically, let $o'' \in \mathcal{O}_B(\mathcal{K})$. Then $n_1(o'') := \mathit{op}^B(\mathit{unfold}(\mathcal{K}''))$, where $\mathcal{K}''$ denotes the unique subbraid of $\mathcal{K}$ with root $c := \pi(o'')$. Now note that in this situation

$$n_1(d) = \mathit{pur}^\Delta(\mathit{unfold}(\mathcal{K}')),$$
$$m_1(c) = \mathit{pur}^\Sigma(\mathit{unfold}(\mathcal{K}'')).$$

To see the first equality, recall that $n_1(d)$ is obtained by replacing each open atom o_r of the root d of $\mathcal{K}'$ by the open atom $\mathit{op}^B(\mathit{unfold}(\mathcal{K}_r))$ where $\mathcal{K}_r$ is the subbraid with root $\pi(o_r)$. On the other hand, when we unfold $\mathcal{K}'$, then the maximal Σ-subtrees are exactly the trees of the form $\mathit{unfold}(\mathcal{K}_r)$. Purification replaces these subtrees by the open atoms $\mathit{op}^B(\mathit{unfold}(\mathcal{K}_r))$, which shows that $\mathit{pur}^\Delta(\mathit{unfold}(\mathcal{K}')) = n_1(d)$. The second equality follows in the same way.

It is now simple to prove that (m_1, n_1) is a simplifier for $\mathcal{K}$: assume the $m_1(o_1) = m_1(o_2)$ for $o_1, o_2 \in \mathcal{O}_A(\mathcal{K})$. For $i = 1, 2$, let $\mathcal{K}_i$ denote that subbraid of $\mathcal{K}$ with root $d_i := \pi(o)$. Then $\mathit{op}^A(\mathit{unfold}(\mathcal{K}_1)) = \mathit{op}^A(\mathit{unfold}(\mathcal{K}_2))$. Since the mapping op^A is injective, $\mathit{unfold}(\mathcal{K}_1) = \mathit{unfold}(\mathcal{K}_2)$. The first of the two equalities given above shows that $n_1(d_1) = \mathit{pur}^\Delta(\mathit{unfold}(\mathcal{K}_1)) = \mathit{pur}^\Delta(\mathit{unfold}(\mathcal{K}_2)) = n_1(d_2)$. With a symmetrical argument, as usual, it follows that (m_1, n_1) is a simplifier for $\mathcal{K}$.

In order to show that the step from $\mathcal{K}$ to $\mathit{unfold} \circ \mathit{fold}^*(\mathcal{K})$ can be described as a simplification of $\mathcal{K}$ it remains to prove that $\mathit{fold}^*(\mathit{unfold}(\mathcal{K})) = \mathcal{K}^{\langle m_1, n_1 \rangle}$.

We first show that both braids have the same root. Without loss of generality, $\mathcal{K}$ has type A. Let a be the root of $\mathcal{K}$. The root of $\mathit{fold}^*(\mathit{unfold}(\mathcal{K}))$ is $\mathit{pur}^\Sigma(\mathit{unfold}(\mathcal{K}))$. The root of $\mathcal{K}^{\langle m_1, n_1 \rangle}$ is $m_1(a)$. Both roots are identical, by the second of the two equalities given above.

We now show that the fibering functions of the two braids coincide on the set of common open atoms. From this it follows easily that both braids are identical

(see Lemma 8.2 for a formal proof). Assume that $o' \in \mathcal{O}_A(\mathit{fold}^*(\mathit{unfold}(\mathcal{K}))) \cap \mathcal{O}_A(\mathcal{K}^{\langle m_1,n_1\rangle})$. Since $o' \in \mathcal{O}_A(\mathcal{K}^{\langle m_1,n_1\rangle})$ we know that o' has the form $m_1(o)$, where $o \in \mathcal{O}_A(\mathcal{K})$. Let $\mathcal{K}_1$ denote the subbraid of $\mathcal{K}$ with root $d_1 := \pi(o)$. Then the fibering function of $\mathcal{K}^{\langle m_1,n_1\rangle}$ maps $o' = m_1(o)$ to the element $n_1(d_1)$. On the other hand $m_1(o) = \mathit{op}^A(\mathit{unfold}(\mathcal{K}_1))$, and the fibering function of $\mathit{fold}^*(\mathit{unfold}(\mathcal{K}))$ maps $m_1(o)$ to the element $\mathit{pur}^\Delta(\mathit{unfold}(\mathcal{K}_1))$. The first of the two equalities given above shows that o' is fibered with the same element $n_1(d_1) = \mathit{pur}^\Delta(\mathit{unfold}(\mathcal{K}_1))$ in both braids. The same holds of course for the open atoms in $\mathcal{O}_B(\mathit{fold}^*(\mathit{unfold}(\mathcal{K}))) \cap \mathcal{O}_B(\mathcal{K}^{\langle m_1,n_1\rangle})$.

Summarizing, we have seen that the step from $\mathcal{K}$ to $\mathit{unfold} \circ \mathit{fold}^*(\mathcal{K})$ can be described as a simplification of $\mathcal{K}$, which—as we have seen already—shows that *fold* and *unfold* are inverse bijections. In the sequel we may assume without loss of generality that op^A and op^B are the mappings that assigns to each rational tree $t \in R(\Sigma \cup \Delta, Z)$ the open atom $o_{\mathit{fold}(t)}$.

Part 4. Here we want to show that *fold* and *unfold* are homomorphisms. Let $\mathcal{K} \in R(\Sigma, X) \odot R(\Delta, Y)$ be a trivial braid, or a nontrivial braid of type A. The element $\mathit{unfold} \circ \mathit{pur}^\Sigma(\mathcal{K})$ is the root of $\mathit{unfold} \circ \mathit{fold}(\mathcal{K})$, i.e., the root of $\mathcal{K}$. Hence $\mathit{unfold} \circ \mathit{pur}^\Sigma(\mathcal{K}) = \mathit{root}_A(\mathcal{K})$, by the definition of root_A. Next assume that $\mathcal{K} \in R(\Sigma, X) \odot R(\Delta, Y)$ is a nontrivial braid of type B. Then $\mathit{unfold} \circ \mathit{pur}^\Sigma(\mathcal{K}) = \mathit{op}^A(\mathit{unfold}(\mathcal{K})) = o_{\mathit{fold}(\mathit{unfold}(\mathcal{K}))} = o_\mathcal{K} = \mathit{root}_A(\mathcal{K})$. We have seen that $\mathit{root}_A = \mathit{unfold} \circ \mathit{pur}^\Sigma$. Similarly it follows that $\mathit{root}_B = \mathit{unfold} \circ \mathit{pur}^\Delta$. Hence $\mathit{pur}^\Sigma = \mathit{fold} \circ \mathit{root}_A$ and $\mathit{pur}^\Delta = \mathit{fold} \circ \mathit{root}_B$. Let $f \in \Sigma$ be n-ary, let $\mathcal{K}_1, \ldots, \mathcal{K}_n \in R(\Sigma, X) \odot R(\Delta, Y)$. Then

$$
\begin{aligned}
&\mathit{unfold}(f_{\mathcal{A}\odot\mathcal{B}}(\mathcal{K}_1, \ldots, \mathcal{K}_n)) = \\
&\mathit{unfold}(\mathit{root}_A^{-1}(f_{\mathcal{A}_*}(\mathit{root}_A(\mathcal{K}_1), \ldots, \mathit{root}_A(\mathcal{K}_n)))) = \\
&\mathit{unfold}(\mathit{root}_A^{-1}(f(\mathit{pur}^\Sigma(\mathit{unfold}(\mathcal{K}_1)), \ldots, \mathit{pur}^\Sigma(\mathit{unfold}(\mathcal{K}_n))))) = \\
&\mathit{unfold}(\mathit{root}_A^{-1}(\mathit{pur}^\Sigma(f(\mathit{unfold}(\mathcal{K}_1), \ldots, \mathit{unfold}(\mathcal{K}_n))))) = \\
&\mathit{unfold}(\mathit{root}_A^{-1}(\mathit{root}_A(\mathit{fold}(f(\mathit{unfold}(\mathcal{K}_1), \ldots, \mathit{unfold}(\mathcal{K}_n)))))) = \\
&f(\mathit{unfold}(\mathcal{K}_1), \ldots, \mathit{unfold}(\mathcal{K}_n)).
\end{aligned}
$$

It follows that *unfold* is a Σ-homomorphism. In the same way it follows that *unfold* is a Δ-homomorphism. It is then trivial to see that *fold* is $(\Sigma \cup \Delta)$-homomorphic, too. This completes the proof of Part 4 and of Proposition 7.3. □

8 Mathematical Background

In this section we study braids, in particular simplification of braids, from a mathematical point of view. On this basis, formal proofs for the statements

of earlier sections are given. As in Section 4, $(\mathcal{A}^{\Sigma}, \mathcal{M}, X)$ and $(\mathcal{B}^{\Delta}, \mathcal{N}, Y)$ denote two fixed non-collapsing SC-structures over disjoint signatures. Our first lemma is a trivial consequence of the definition of admissible endomorphisms.

Lemma 8.1 *Let $C \subseteq A$. If the admissible endomorphisms $m_1, m_2 \in \mathcal{M}$ coincide on $\mathcal{O}_A(A')$, then m_1 and m_2 coincide on A'. Similarly, let $D \subseteq B$. If the admissible endomorphisms $n_1, n_2 \in \mathcal{N}$ coincide on $\mathcal{O}_B(D)$, then n_1 and n_2 coincide on D.*

8.1 Subbraids and variants

We first collect some simple facts about braid forests, variants, and their subbraids. The reader should recall the details of the definition of a braid (forest) given in Definition 4.4. We write $\mathcal{K}' \subseteq \mathcal{K}$ if $\mathcal{K}'$ is a subbraid of the braid (forest) $\mathcal{K}$.

Lemma 8.2 *For $i = 1, 2$, let $\mathcal{K}_i$ a braid with root a_i and fibering function π_i. If $a_1 = a_2$, and if $\pi_1(o) = \pi_2(o)$ for all $o \in \mathcal{O}(\mathcal{K}_1) \cap \mathcal{O}(\mathcal{K}_2)$, then $\mathcal{K}_1 = \mathcal{K}_2$.*

Proof. Let e_0 be an element of $\mathcal{K}_1$. Then there exists a sequence $e = e_0, \ldots, e_n = a_1$ of elements of $\mathcal{K}_1$ such that every e_i is directly fibered via π_1 with e_{i+1}, for $0 = 1, \ldots, n-1$. A simple induction on the length n using the assumptions of the lemma shows that e_0 is an element of $\mathcal{K}_2$. With a symmetric argument it follows that $\mathcal{K}_1$ and $\mathcal{K}_2$ have the same elements. Obviously this implies that $\mathcal{K}_1 = \mathcal{K}_2$. □

Corollary 8.3 *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be two braid forests and $\mathcal{K}_1 \subseteq \mathcal{K}_2$. Let $\mathcal{K}_1'$ be a subbraid of $\mathcal{K}_1$ and let $\mathcal{K}_2'$ be a subbraid of $\mathcal{K}_2$ such that $\mathcal{K}_1'$ and $\mathcal{K}_2'$ have the same root. Then $\mathcal{K}_1' = \mathcal{K}_2'$.*

Proof. Since $\mathcal{K}_1'$ and $\mathcal{K}_2'$ are subbraids of $\mathcal{K}_2$ it is obvious that $\mathcal{K}_1'$ and $\mathcal{K}_2'$ satisfy the conditions of Lemma 8.2. Hence $\mathcal{K}_1 = \mathcal{K}_2$. □

Lemma 8.4 *For each element e of a braid forest $\mathcal{K}$ there exists a unique subbraid $\mathcal{K}'$ of $\mathcal{K}$ with root e. The elements of $\mathcal{K}'$ are the elements of $\mathcal{K}$ that are fibered with e in $\mathcal{K}$.*

Proof. It is straightforward to verify that the elements of $\mathcal{K}$ that are fibered with e form a subbraid of $\mathcal{K}$ with root e. By Corollary 8.3 it is the unique subbraid of $\mathcal{K}$ with root e. □

Lemma 8.5 *If two braid forests are variants, then the two subbraids given by their roots are variants. Each variant of a braid is a braid.*

Proof. Let $\mathcal{K}$ be a braid forest with root a and fibering function π, let (m, n) be an admissible pair of automorphisms, and let $\mathcal{K}'$ be the variant of $\mathcal{K}$ that is defined as in Definition 4.7. Using the second part of Lemma 3.10 it is easy to show that two elements of $\mathcal{K}$ are fibered in $\mathcal{K}$ if and only if their images under m and n respectively are fibered in $\mathcal{K}'$. It is then simple to verify that the subbraid of $\mathcal{K}'$ given by the root of $\mathcal{K}'$ is the image of the unique subbraid of $\mathcal{K}$ with root a under (m, n). This proves the first part. If $\mathcal{K}$ is a braid with root a, say, of type A, then the above argument shows that each element of the variant $\mathcal{K}'$ is fibered with the new root $m(a)$. It follows that $\mathcal{K}'$ is a braid. □

For the next lemma, note that we presuppose in Definition 4.7 that both $\mathcal{K}$ and $\mathcal{K}'$ are braids (braid forests). In the following lemma, we do not make this assumption on $\mathcal{K}'$.

Lemma 8.6 *Let (m, n) be an admissible pair of automorphisms. Let $\mathcal{K}$, a', C', D', π'_A, and π'_B be defined as in Definition 4.7, 1.-3. Then $\mathcal{K}' := \langle a', C', D', \pi'_A, \pi'_B \rangle$ is a braid (forest) and a variant of $\mathcal{K}$.*

Proof. Since m and n are admissible automorphisms it is straightforward to verify that $\mathcal{K}'$ satisfies Conditions 1–3 of Definition 4.4. Thus $\mathcal{K}'$ is a braid forest. Clearly it is a variant of $\mathcal{K}$. □

8.2 Simplification of braid forests and braids

Simplification of braids turns out to be technically difficult. Let us start with two simple facts. The proof of the first lemma is straightforward.

Lemma 8.7 *Let (m, n) be a simplifier for the braid forest $\mathcal{K}$. Then (m, n) is a simplifier for each subforest of $\mathcal{K}$.*

We may now prove

Lemma 4.20 *Let (m, n) be a simplifier for the braid forest $\mathcal{K}$, let $\mathcal{K}_1$ be the unique subbraid of $\mathcal{K}$ with root e, where e is an element of $\mathcal{K}$ of type A (resp. B). Then $\mathcal{K}_1^{\langle m,n \rangle}$ is the unique subbraid of $\mathcal{K}^{(m,n)}$ with root $m(e)$ (resp. $n(e)$).*

Proof. It follows directly from Definition 4.9 that $\mathcal{K}_1^{\langle m,n \rangle}$ is a subbraid of the braid forest $\mathcal{K}^{(m,n)}$. Obviously $m(e)$ (resp. $n(e)$) is the root of $\mathcal{K}_1^{\langle m,n \rangle}$. Now use Corollary 8.3. □

There is one technical point behind the definition of a simplifier that will cause some difficulties in the further development. Assume, in the situation of Definition 4.9, that $\mathcal{O}_A(C) = \{o_1, \ldots, o_k\}$ and $\mathcal{O}_B(D) = \{u_1, \ldots, u_l\}$. Then there is no guarantee that $\mathcal{O}_A(C') = \{m(o_1), \ldots, m(o_k)\}$ and $\mathcal{O}_B(D') =$

$\{n(u_1), \ldots, n(u_l)\}$. In fact, Lemma 3.10 only shows the inclusion $\mathcal{O}_A(C') \subseteq \{m(o_1), \ldots, m(o_k)\}$.

Definition 8.8 The set

$$(\{m(o) \mid o \in \mathcal{O}_A(C)\} \setminus \mathcal{O}_A(C')) \ \cup \ (\{n(o) \mid o \in \mathcal{O}_B(D)\} \setminus \mathcal{O}_B(D'))$$

is called the set of *lost atoms* of the simplification step leading from $\mathcal{K}$ to $\mathcal{K}^{(m,n)}$.

The phenomenon that atoms may be lost at a simplification step is responsible for the problem that in general $\mathcal{K}^{(m,n)} \neq \mathcal{K}^{\langle m,n\rangle}$ (cf. Def. 4.9). In principle we could restrict the amalgamation construction to a class of structures for which we can replace the inclusion given in Lemma 3.10 by an equality. In this case, lost atoms could not occur and simplification would become much simpler. However, our motivation was to give a general construction. For this reason we shall not follow this line. In the following discussion we shall use the notation introduced in Definition 4.9. The following lemma follows immediately from the definition of $\mathcal{K}^{(m,n)}$.

Lemma 8.9 *Assume, in the situation of Definition 4.9, that $o \in \mathcal{O}_A(C)$ and $m(o)$ is not a lost atom of the simplification step leading from $\mathcal{K}$ to $\mathcal{K}^{(m,n)}$. Then $m(o) \in \mathcal{O}_A(C')$ and $\pi'_A(m(o)) = n(\pi_A(o))$.*

If (m, n) is a simplifier for the braid forest $\mathcal{K}$, the step from $\mathcal{K}$ to $\mathcal{K}^{(m,n)}$ will be called simplification of the braid forest $\mathcal{K}$. While we are mainly interested in simplification of braids (where we have to use $\mathcal{K}^{\langle m,n\rangle}$), it turns out to be simpler to treat simplification of braid forests before.

Lemma 8.10 *Let (m_1, n_1) be a simplifier for the braid forest $\mathcal{K}_0$, and let (m_2, n_2) be a simplifier for the braid forest $\mathcal{K}_1 = \mathcal{K}_0^{(m_1,n_1)}$. Assume that m_2 and n_2 do not identify any lost atom of the simplification step leading from $\mathcal{K}_0$ to $\mathcal{K}_1$ with another atom. Then $(m_1 \circ m_2, n_1 \circ n_2)$ is a simplifier for $\mathcal{K}_0$ and $\mathcal{K}_0^{(m_1 \circ m_2, n_1 \circ n_2)} = \mathcal{K}_1^{(m_2,n_2)}$.*

Proof. Let $\mathcal{K}_i = \langle a_i, C_i, D_i, \pi^i_A, \pi^i_B\rangle$, for $i = 0, 1$. We may assume that both are of type A. Let $(m, n) := (m_1 \circ m_2, n_1 \circ n_2)$. If $m(o) = m(o')$ for $o, o' \in \mathcal{O}_A(C_0)$, then either $m_1(o) = m_1(o')$, or $m_1(o) \neq m_1(o')$ and $m_2(m_1(o)) = m_2(m_1(o'))$. In the former case we know that $n_1(\pi^0_A(o)) = n_1(\pi^0_A(o'))$ since (m_1, n_1) is a simplifier for $\mathcal{K}_0$. Hence $n(\pi^0_A(o)) = n(\pi^0_A(o'))$. In the latter case, neither $m_1(o)$ nor $m_1(o')$ can be lost, by assumption. Hence, by Lemma 8.9, $m_1(o)$ and $m_1(o')$ are in $\mathcal{O}_A(\mathcal{K}_1)$, and moreover $n_1(\pi^0_A(o)) = \pi^1_A(m_1(o))$ and $n_1(\pi^0_A(o')) = \pi^1_A(m_1(o'))$. Since $m_2(m_1(o)) = m_2(m_1(o'))$ and (m_2, n_2) is a simplifier for $\mathcal{K}_1$, this implies that

$$n_2(n_1(\pi^0_A(o))) = n_2(\pi^1_A(m_1(o))) = n_2(\pi^1_A(m_1(o'))) = n_2(n_1(\pi^0_A(o'))).$$

Hence in both cases $n(\pi^0_A(o)) = n(\pi^0_A(o'))$. Symmetrically it follows that $n(o) = n(o')$ implies $m(\pi^0_B(o)) = m(\pi^0_B(o'))$, for all $o, o' \in \mathcal{O}_A(D_0)$. Hence (m, n) is a simplifier for $\mathcal{K}_0$. It follows directly from Definition 4.9 that $\mathcal{K}_1^{(m_2,n_2)} = (\mathcal{K}_0^{(m_1,n_1)})^{(m_2,n_2)} = \mathcal{K}_0^{(m_1 \circ m_2, n_1 \circ n_2)}$. □

Corollary 8.11 *Let (m_1, n_1) be a simplifier for the braid forest $\mathcal{K}_0$, and let (m_2, n_2) be a simplifier for the braid forest $\mathcal{K}_1 = \mathcal{K}_0^{(m_1,n_1)}$. Then there exists a simplifier (m, n) for $\mathcal{K}_0$ such that $\mathcal{K}_0^{(m,n)} = \mathcal{K}_1^{(m_2,n_2)}$.*

Proof. It follows from Lemma 8.1 that there exists a simplifier (m'_2, n'_2) of $\mathcal{K}_1$ such that (m'_2, n'_2) does not identify any lost atom of the simplification step leading from $\mathcal{K}_0$ to $\mathcal{K}_1$ with another atom, and $\mathcal{K}_1^{(m_2,n_2)} = \mathcal{K}_1^{(m'_2,n'_2)}$. Let $(m, n) := (m_1 \circ m'_2, n_1 \circ n'_2)$. Then, by the previous lemma, $\mathcal{K}_0^{(m,n)} = \mathcal{K}_1^{(m'_2,n'_2)} = \mathcal{K}_1^{(m_2,n_2)}$. □

Lemma 8.12 *Let (m, n) be a simplifier for the braid (forest) $\mathcal{K}$. If $|\mathcal{O}(\mathcal{K})| = |\mathcal{O}(\mathcal{K}^{(m,n)})|$, then $\mathcal{K}$ and $\mathcal{K}^{(m,n)}$ are variants.*

Proof. Under the given assumption it follows from Lemma 3.10 that the restrictions of m and n on $\mathcal{O}_A(\mathcal{K})$ and $\mathcal{O}_B(\mathcal{K})$ respectively are injective. Therefore there exists an admissible pair of automorphisms (m', n') such that m and m' coincide on $\mathcal{O}_A(\mathcal{K})$, and similarly n and n' coincide on $\mathcal{O}_B(\mathcal{K})$. Now the result follows from Lemma 8.6 and Lemma 8.1. □

The previous lemma focussed on $\mathcal{K}^{(m,n)}$. Recall that the image of $\mathcal{K}$ under (m, n) is $\mathcal{K}^{\langle m,n \rangle}$ (cf. Def. 4.9). The corresponding lemma for $\mathcal{K}^{\langle m,n \rangle}$ was mentioned in Section 4.

Lemma 4.10 *Let (m, n) be a simplifier for the braid $\mathcal{K}$. If $|\mathcal{O}(\mathcal{K})| = |\mathcal{O}(\mathcal{K}^{\langle m,n \rangle})|$, then $\mathcal{K}$ and $\mathcal{K}^{\langle m,n \rangle}$ are variants.*

Proof. $|\mathcal{O}(\mathcal{K})| = |\mathcal{O}(\mathcal{K}^{\langle m,n \rangle})|$ implies $|\mathcal{O}(\mathcal{K})| = |\mathcal{O}(\mathcal{K}^{(m,n)})|$. The previous lemma shows that $\mathcal{K}^{(m,n)}$ is a variant of $\mathcal{K}$. Part 2 of Lemma 8.5 shows that $\mathcal{K}^{(m,n)}$ is a braid. Clearly this implies that $\mathcal{K}^{(m,n)} = \mathcal{K}^{\langle m,n \rangle}$. □

Similarly as for braids, call a simplifier (m, n) of a braid forest $\mathcal{K}$ *proper* if $|\mathcal{O}(\mathcal{K})| > |\mathcal{O}(\mathcal{K}^{(m,n)})|$. Obviously we have

Lemma 8.13 *$|\mathcal{O}(\mathcal{K})|$ gives an upper bound on the length of every sequence of proper simplifications for the braid forest $\mathcal{K}$.*

A braid forest $\mathcal{K}'$ is called *irreducible* if $\mathcal{K}'$ does not have a strict simplifier. We want to show that all irreducible braid forests that can be reached from a braid forest $\mathcal{K}$ by simplification are variants. For this purpose, the following lemma is needed that shows that simplification of braid forests is "locally confluent".

Lemma 8.14 *Let (m_1, n_1) and (m_2, n_2) be two simplifiers for the braid forest $\mathcal{K}_0$, let $\mathcal{K}_1$ and $\mathcal{K}_2$ be the images of $\mathcal{K}_0$ under (m_1, n_1) and (m_2, n_2) respectively. Then there exist a simplifier (m_3, n_3) for $\mathcal{K}_1$ and a simplifier (m_4, n_4) for $\mathcal{K}_2$ such that $\mathcal{K}_1^{(m_3,n_3)} = \mathcal{K}_2^{(m_4,n_4)}$.*

Proof. Let $\mathcal{K}_i = \langle a_i, C_i, D_i, \pi_A^i, \pi_B^i \rangle$, for $i = 0, 1, 2$. The endomorphisms m_1 and n_1 define equivalence relations $\sim_A^1$ and $\sim_B^1$ on $\mathcal{O}_A(C_0)$ and $\mathcal{O}_B(D_0)$ respectively, where elements are equivalent with respect to $\sim_A^1$ ($\sim_B^1$) iff they have the same image under m_1 (n_1). The endomorphisms m_2 and n_2 define similar equivalence relations $\sim_A^2$ and $\sim_B^2$ on $\mathcal{O}_A(C_0)$ and $\mathcal{O}_B(D_0)$ respectively. Let $\sim_A := \sim_A^1 \sqcup \sim_A^2 = (\sim_A^1 \cup \sim_A^2)^*$ denote the smallest equivalence relation on $\mathcal{O}_A(\mathcal{K}_0)$ that extends $\sim_A^1$ and $\sim_A^2$. Similarly, let $\sim_B$ denote the smallest equivalence relation on $\mathcal{O}_B(\mathcal{K}_0)$ that extends $\sim_B^1$ and $\sim_B^2$. Choose a system of representants for $\sim_A$ and a similar system for $\sim_B$. We write $rep(o)$ for the representant of the equivalence class $[o]$ of $o \in \mathcal{O}_A(\mathcal{K})$ ($o \in \mathcal{O}_B(\mathcal{K})$) with respect to $\sim_A$ ($\sim_B$).

The elements of $\mathcal{O}_A(C_1)$ have the form $m_1(o_A)$ for $o_A \in \mathcal{O}_A(C_0)$, by Lemma 3.10. If, for $o_A, o'_A \in \mathcal{O}_A(C_0)$, $m_1(o_A) = m_1(o'_A) \in \mathcal{O}_A(C_1)$, then $o_A \sim_A^1 o'_A$ and $rep(o_A) = rep(o'_A)$. Thus

- the mapping $m_1(o_A) \mapsto rep(o_A)$ ($o_A \in \mathcal{O}_A(C_0)$) is welldefined. It can be extended to an admissible endomorphism $m_3 \in \mathcal{M}$. Similarly the mapping $n_1(o_B) \mapsto rep(o_B)$ ($o_B \in \mathcal{O}_B(D_0)$) is welldefined and can be extended to an admissible endomorphism $n_3 \in \mathcal{N}$.

Symmetrically we can show

- the mapping $m_2(o_A) \mapsto rep(o_A)$ ($o_A \in \mathcal{O}_A(C_0)$) is welldefined and can be extended to an admissible endomorphism $m_4 \in \mathcal{M}$, and the mapping $n_2(o_B) \mapsto rep(o_B)$ ($o_B \in \mathcal{O}_B(D_0)$) is welldefined and can be extended to an admissible endomorphism $n_4 \in \mathcal{N}$.

We have $(*)$

$$\begin{aligned} m_3(m_1(o_A)) = rep(o_A) = m_4(m_2(o_A)) \qquad (o_A \in \mathcal{O}_A(C_0)) \\ n_3(n_1(o_B)) = rep(o_B) = n_4(n_2(o_B)) \qquad (o_B \in \mathcal{O}_B(D_0)) \end{aligned}$$

and, by Lemma 8.1, $(**)$

$$\begin{aligned} m_3(m_1(c)) = m_4(m_2(c)) \qquad (c \in C_0) \\ n_3(n_1(d)) = n_4(n_2(d)) \qquad (d \in D_0). \end{aligned}$$

Clearly (m_3, n_3) and (m_4, n_4) are admissible. We now show that (m_3, n_3) is a simplifier for $\mathcal{K}_1$. Let $\langle o'_1, b'_1 \rangle, \langle o'_2, b'_2 \rangle \in \pi_A^1$ and suppose that $m_3(o'_1) = m_3(o'_2)$. We have to verify that $n_3(b'_1) = n_3(b'_2)$. For $i = 1, 2$, there exists $o_i \in \mathcal{O}_A(C_0)$

and $b_i := \pi_A^0(o_i) \in D_0$ such that $o_i' = m_1(o_i)$ and $b_i' = n_1(b_i)$. Since m_3 identifies o_1' and o_2' we know that $o_1 \sim_A o_2$. Thus there exists a sequence $o_1 = u_1, u_2 \ldots, u_k = o_2$ such that each pair $\langle u_i, u_{i+1}\rangle$ belongs either to $\sim_A^1$ or to $\sim_A^2$ $(1 \leq i < k)$. Let $d_i := \pi_A^0(u_i)$, for $i = 1, \ldots, k$. Thus $d_i \in D_0$ $(1 \leq i \leq k)$ and we have $b_1 = d_1$ and $b_2 = d_k$. Now (m_1, n_1) and (m_2, n_2) are simplifiers. Thus, if $\langle u_i, u_{i+1}\rangle \in \sim_A^1$, then $n_1(d_i) = n_1(d_{i+1})$, which implies $n_3(n_1(d_i)) = n_3(n_1(d_{i+1}))$, and if $\langle u_i, u_{i+1}\rangle \in \sim_A^2$, then $n_2(d_i) = n_2(d_{i+1})$, which implies $n_4(n_2(d_i)) = n_4(n_2(d_{i+1}))$ and, by $(**)$, $n_3(n_1(d_i)) = n_3(n_1(d_{i+1}))$. Therefore we obtain

$$n_3(b_1') = n_3(n_1(b_1)) = n_3(n_1(b_2)) = n_3(b_2')$$

as desired. We have shown that (m_3, n_3) is a simplifier for $\mathcal{K}_1$. Symmetrically it follows that (m_4, n_4) is a simplifier for $\mathcal{K}_2$.

By $(**)$, the two braid forests $\mathcal{K}_1^{(m_3,n_3)}$ and $\mathcal{K}_2^{(m_4,n_4)}$ have the same root $m_3(m_1(a_0))$. Moreover, for both forests the set $\{m_3(m_1(c)) \mid c \in C_0\}$ gives the set of elements of type A, $\{n_3(n_1(d)) \mid d \in D_0\}$ is the set of elements of type B, and

$$\{\langle m_3(m_1(o)), n_3(n_1(d))\rangle \mid \langle o, d\rangle \in \pi_A^0\} \cup \{\langle n_3(n_1(o)), m_3(m_1(c))\rangle \mid \langle o, c\rangle \in \pi_B^0\}$$

is the fibering function. This shows that $\mathcal{K}_1^{(m_3,n_3)}$ and $\mathcal{K}_2^{(m_4,n_4)}$ are identical. □

Theorem 8.15 *Each sequence of iterated proper simplifications that starts from the braid forest $\mathcal{K}$ has length $\leq |\mathcal{O}(\mathcal{K})|$. If $\mathcal{K}'$ is an irreducible braid forest that is obtained from $\mathcal{K}$ by a sequence of simplifications, then there exists a simplifier (m, n) for $\mathcal{K}$ such that $\mathcal{K}^{(m,n)} = \mathcal{K}'$. If two irreducible braid forests $\mathcal{K}_1$ and $\mathcal{K}_2$ can be reached from $\mathcal{K}$ by sequences of simplifications, then $\mathcal{K}_1$ and $\mathcal{K}_2$ are variants.*

Proof. The first statement is Lemma 8.13. The second statement follows from Corollary 8.11 with a trivial induction. If $\mathcal{K}_1$ and $\mathcal{K}_2$ are two irreducible braid forests that are obtained from $\mathcal{K}$ by sequences of simplifications, then both braid forests can be obtained from $\mathcal{K}$ by a single simplification step, by Corollary 8.11. Lemma 8.14 shows that there exists a braid forest $\mathcal{K}_3$ that can be reached from $\mathcal{K}_1$ and $\mathcal{K}_2$ by simplification. Since $\mathcal{K}_1$ and $\mathcal{K}_2$ are irreducible, these simplification steps are not proper. Hence $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$ are variants, by Lemma 8.12. □

Before we treat simplification of braids, let us mention three properties of irreducible braid forests.

Lemma 8.16 *(a) If the braid forest $\mathcal{K} = \langle a, C, D, \pi_A, \pi_B\rangle$ is irreducible, then π_A and π_B are injective.*
(b) If $\mathcal{K}'$ is a subbraid of the irreducible braid forest $\mathcal{K}$, then $\mathcal{K}'$ is irreducible.

(c) If $\mathcal{K}_1$ and $\mathcal{K}_2$ are subbraids of the irreducible braid forest $\mathcal{K}$, and if $\mathcal{K}_1$ and $\mathcal{K}_2$ are variants, then $\mathcal{K}_1 = \mathcal{K}_2$.

Proof. (a) Assume that π_A, say, is not injective. Then there exist elements $\langle o_1, b_1\rangle$ and $\langle o_2, b_1\rangle$ in π_A where o_1 and o_2 are distinct. Let $m \in \mathcal{M}$ be an admissible endomorphism that maps o_1 to o_2 and leaves all other atoms fixed, let n be the identity on B. Now (m, n) is a strict simplifier for $\mathcal{K}$, thus we get a contradiction.

(b) Assume, to get a contradiction, that (m, n) is a proper simplifier for $\mathcal{K}'$. Let $\mathcal{K} = \langle a, C, D, \pi_A, \pi_B\rangle$, let $\mathcal{K}' = \langle a', C', D', \pi'_A, \pi'_B\rangle$. Let $X_A = \mathcal{O}_A(C) \setminus \mathcal{O}_A(C')$, let $Y_B = \mathcal{O}_B(D) \setminus \mathcal{O}_B(D')$. By Lemma 8.1 we may assume that m (n) leaves the elements of X_A (Y_B) fixed.

If the sets $\{m(o) \mid o \in \mathcal{O}_A(C')\} \cap X_A$ and $\{n(o) \mid o \in \mathcal{O}_B(D')\} \cap Y_B$ both are empty, then any pair of open atoms that are identified by m (n) belongs to $\mathcal{O}(\mathcal{K}')$. In this case it is easy to see that (m, n) is a proper simplifier for $\mathcal{K}$, which yields a contradiction.

In the other case, let m' and n' be admissible automorphisms such that the sets $\{m'(m(o)) \mid o \in \mathcal{O}_A(C')\} \cap X_A$ and $\{n'(n(o)) \mid o \in \mathcal{O}_B(D')\} \cap Y_B$ are empty. Let m^* denote the endomorphism that coincides with $m \circ m'$ on $\mathcal{O}_A(C')$ and leaves all other open atoms fixed. Let n^* denote the endomorphism that coincides with $n \circ n'$ on $\mathcal{O}_B(D')$ and leaves all other open atoms fixed. Then (m^*, n^*) is a proper simplifier for $\mathcal{K}$, which yields a contradiction.

(c) Let $\mathcal{K} = \langle a, C, D, \pi_A, \pi_B\rangle$, let $\mathcal{K}_i = \langle a_i, C_i, D_i, \pi^i_A, \pi^i_B\rangle$ $(i = 1, 2)$. Assume that $\mathcal{K}_1$ and $\mathcal{K}_2$ are variants, but $\mathcal{K}_1 \neq \mathcal{K}_2$. There exists a pair of admissible automorphisms (m, n) such that $\mathcal{K}_2 = \mathcal{K}_1^{(m,n)}$. Without loss of generality we have ($\diamond$): there exists an $o^* \in \mathcal{O}_A(C_1)$ such that $o^* \neq m(o^*) \in \mathcal{O}_A(C_2)$.

Consider an element $o^A_0 \in \mathcal{O}_A(C_1)$. If all elements of the "orbit"

$$o^A_0,\ o^A_1 := m(o^A_0),\ o^A_2 := m(o^A_1),\ o^A_3 := m(o^A_2)\ldots$$

are in $\mathcal{O}_A(C_1)$, then this sequence contains only a finite number of distinct elements, say, $o^A_0, \ldots, o^A_k$. In the other case, let k be the first index in the sequence $0, 1, \ldots$ such that $o^A_k \notin \mathcal{O}_A(C_1)$. This implies that $o^A_k \in \mathcal{O}_A(C_2)$. The set $\{o^A_0, \ldots, o^A_k\}$ is called the m-trace $tr_m(o^A_0)$ of o^A_0. Let $\sim_m$ be the smallest equivalence relation on $\mathcal{O}_A(C_1) \cup \mathcal{O}_A(C_2)$ such that $o \sim_m o'$ whenever o and o' both belong to the m-trace $tr(o^A)$ of the same element $o^A \in \mathcal{O}_A(C_1)$. Since m is an injective function, the equivalence classes of $\sim_m$ are just the maximal m-traces. For each equivalence class $[o^A]_{\sim_m}$, choose a representant $rep([o^A]_{\sim_m})$. Let $m_\infty \in \mathcal{M}$ be the admissible endomorphism that maps each $o^A \in \mathcal{O}_A(C_1) \cup \mathcal{O}_A(C_2)$ to the representant $rep([o^A]_{\sim_m})$ and leaves other atoms fixed. Since $o^A \in \mathcal{O}_A(C_1)$ implies that $m(o^A) \in \mathcal{O}_A(C_2)$, and since both atoms have the same representant, we know that $m_\infty(m(o^A)) = m_\infty(o^A)$ for all $o^A \in \mathcal{O}_A(C_1)$.

This implies, by Lemma 8.1, that $m_\infty(m(a)) = m_\infty(a)$ for all $a \in C_1$.

Symmetrically, we may define the n-traces $tr_n(o^B)$ of elements $o^B \in \mathcal{O}_B(D_1)$, just by replacing $\mathcal{O}_A(C_i)$ by $\mathcal{O}_B(D_i)$ $(i = 1, 2)$ and m by n. We obtain the equivalence relation $\sim_n$ by "identifying" all elements that belong to the same n-trace $tr_n(o^B)$, for some $o^B \in \mathcal{O}_B(D_1)$. For each equivalence class $[o^B]_{\sim_n}$, choose a representant $rep([o^B]_{\sim_n})$. Let $n_\infty \in \mathcal{N}$ be the admissible endomorphism that maps each $o^B \in \mathcal{O}_B(D_1) \cup \mathcal{O}_B(D_2)$ to the representant $rep([o^B]_{\sim_n})$ and leaves other atoms fixed. We have $n_\infty(n(o^B)) = n_\infty(o^B)$ for all $o^B \in \mathcal{O}_B(D_1)$, and $n_\infty(n(b)) = n_\infty(b)$ for all $b \in D_1$.

We want to show that (m_∞, n_∞) is a simplifier for $\mathcal{K}$. Suppose that $m_\infty(o) = m_\infty(o')$ for open atoms $o \neq o' \in \mathcal{O}_A(\mathcal{K})$. We may assume that there exists a sequence $o = o_0^A, o_1^A, \ldots, o_r^A = o'$ of elements of $\mathcal{O}_A(C_1) \cup \mathcal{O}_A(C_2)$, where at least the elements $o_0^A, \ldots, o_{r-1}^A$ are in $\mathcal{O}_A(C_1)$, such that $o_i^A = m^i(o_0^A)$, for $0 \leq i \leq r$. Let $b_i := \pi_A(o_i^A)$ $(0 \leq i \leq r)$. Note that at least the elements $b_0, \ldots, b_{r-1}$ are in D_1 since $\mathcal{K}_1$ is a subbraid of $\mathcal{K}$ and π_A and π_A^1 coincide on $\mathcal{O}_A(C_1)$. Since $\langle o_0^A, b_0 \rangle \in \pi_A^1$ we know, by choice of (m, n), that $\langle m(o_0^A), n(b_0) \rangle \in \pi_A^2 \subseteq \pi_A$, which means that $b_1 = n(b_0)$. Similarly we see that $b_i = n^i(b_0)$ for $i = 0, \ldots, r$. But then we have

$$n_\infty(b_0) = n_\infty(n(b_0)) = n_\infty(b_1) = \ldots = n_\infty(b_{k-1}) = n_\infty(n(b_{r-1})) = n_\infty(b_r)$$

Thus n_∞ identifies the π_A-images of $o = o_0^A$ and $o' = o_r^A$. Symmetrically, if $n_\infty(o) = n_\infty(o')$ for atoms $o, o' \in \mathcal{O}_B(\mathcal{K})$, then m_∞ identifies the π_B-images of o and o'. Therefore (m_∞, n_∞) is in fact a simplifier for $\mathcal{K}$. But (m_∞, n_∞) is strict, by $(\diamond)$. This is a contradiction. Thus $\mathcal{K}_1 = \mathcal{K}_2$. □

We shall now turn to simplification of braids. Let (m, n) be a simplifier for the braid $\mathcal{K}$. In order to emphasize the (potential) difference between the braid forest $\mathcal{K}^{(m,n)}$ and the braid $\mathcal{K}^{\langle m,n \rangle}$ the latter will be called the *braid image* of $\mathcal{K}$ under (m, n) for the rest of this section. First we shall show that the result of two consecutive simplification steps may be obtained by a single simplification, similarly as for braid forests. We have to adapt the notion of a lost atom to the new situation.

Definition 8.17 Let (m, n) be a simplifier for the braid $\mathcal{K} = \langle a, C, D, \pi_A, \pi_B \rangle$. Let $\mathcal{K}^{\langle m,n \rangle} = \langle a', C', D', \pi'_A, \pi'_B \rangle$. Then the set

$$(\{m(o) \mid o \in \mathcal{O}_A(C)\} \setminus \mathcal{O}_A(C')) \ \cup \ (\{n(o) \mid o \in \mathcal{O}_B(D)\} \setminus \mathcal{O}_B(D'))$$

is called the set of *lost atoms* of the simplification step from $\mathcal{K}$ to the braid image $\mathcal{K}^{\langle m,n \rangle}$.

The set of lost atoms of the simplification step from $\mathcal{K}$ to the *braid* image $\mathcal{K}^{\langle m,n \rangle}$ is a *superset* of the set of lost atoms of the simplification step from $\mathcal{K}$ to the image $\mathcal{K}^{(m,n)}$, but both sets are not necessarily identical.

Lemma 8.18 *Let (m_1, n_1) be a simplifier for the braid $\mathcal{K}_0$, let (m_2, n_2) be a simplifier for its braid image $\mathcal{K}_1 := \mathcal{K}_0^{\langle m_1, n_1 \rangle}$. Assume that m_2 and n_2 do not identify any lost atom of the simplification step leading from $\mathcal{K}_0$ to the braid image $\mathcal{K}_1$ with any other atom. Then $(m_1 \circ m_2, n_1 \circ n_2)$ is a simplifier for $\mathcal{K}_0$ and $\mathcal{K}_0^{\langle m_1 \circ m_2, n_1 \circ n_2 \rangle} = \mathcal{K}_1^{\langle m_2, n_2 \rangle}$.*

Proof. Exactly as in the corresponding proof of Lemma 8.10 it follows that $(m_1 \circ m_2, n_1 \circ n_2)$ is a simplifier for $\mathcal{K}_0$. Our assumptions guarantee that (m_2, n_2) is also a simplifier for $\mathcal{K}_0^{(m_1, n_1)}$ such that m_2 and n_2 do not identify any lost atom of the simplification step leading from $\mathcal{K}_0$ to the image $\mathcal{K}_0^{(m_1, n_1)}$ with another atom. Hence Lemma 8.10 implies that $(\mathcal{K}_0^{(m_1, n_1)})^{(m_2, n_2)} = \mathcal{K}_0^{(m_1 \circ m_2, n_1 \circ n_2)}$. Now $\mathcal{K}_1 = \mathcal{K}_0^{\langle m_1, n_1 \rangle}$ is a subbraid of the braid forest $\mathcal{K}_0^{(m_1, n_1)}$ and both have the same root. Lemma 4.20 shows that $\mathcal{K}_1^{\langle m_2, n_2 \rangle}$ is the unique subbraid of $(\mathcal{K}_0^{(m_1, n_1)})^{(m_2, n_2)} = \mathcal{K}_0^{(m_1 \circ m_2, n_1 \circ n_2)}$ given by its root, namely $\mathcal{K}_0^{\langle m_1 \circ m_2, n_1 \circ n_2 \rangle}$. □

Corollary 8.19 *Let (m_1, n_1) be a simplifier for the braid $\mathcal{K}_0$, let (m_2, n_2) be a simplifier for the braid image $\mathcal{K}_1 = \mathcal{K}_0^{\langle m_1, n_1 \rangle}$. Then there exists a simplifier (m, n) for $\mathcal{K}_0$ such that $\mathcal{K}_0^{\langle m, n \rangle} = \mathcal{K}_1^{\langle m_2, n_2 \rangle}$.*

Proof. It follows from Lemma 8.1 that there exists a simplifier (m_2', n_2') of $\mathcal{K}_1$ such that (m_2', n_2') does not identify any lost atom of the simplification step leading from $\mathcal{K}_0$ to the braid image $\mathcal{K}_1$ with another atom, and $\mathcal{K}_1^{\langle m_2, n_2 \rangle} = \mathcal{K}_1^{\langle m_2', n_2' \rangle}$. Let $(m, n) := (m_1 \circ m_2', n_1 \circ n_2')$. Then, by the previous lemma, $\mathcal{K}_0^{\langle m, n \rangle} = \mathcal{K}_1^{\langle m_2', n_2' \rangle} = \mathcal{K}_1^{\langle m_2, n_2 \rangle}$. □

The following theorem from Section 4 summarizes the most important properties of braid simplification.

Theorem 4.12 *Let $\mathcal{K}$ be a braid.*

(1) Let $\mathcal{K} = \mathcal{K}_0, \mathcal{K}_1, \ldots, \mathcal{K}_k$ be a sequence of braids such that each braid $\mathcal{K}_{i+1}$ is the braid image of $\mathcal{K}_i$ under a proper simplification, for $i = 0, \ldots, k-1$. Then $k \leq |\mathcal{O}(\mathcal{K})|$.

(2) If $\mathcal{K}_1$ is irreducible and can be simplified to the braid $\mathcal{K}_2$, then $\mathcal{K}_2$ is irreducible and a variant of $\mathcal{K}_1$.

(3) $\mathcal{K}$ can be simplified to an irreducible braid.

(4) If $\mathcal{K}'$ is an irreducible braid that is reached from $\mathcal{K}$ by a sequence of consecutive simplification steps (always taking braid images), then there exists a simplifier (m, n) for $\mathcal{K}$ such that $\mathcal{K}^{\langle m, n \rangle} = \mathcal{K}'$.

(5) If two irreducible braids $\mathcal{K}_1$ and $\mathcal{K}_2$ can be reached from $\mathcal{K}$ by sequences of consecutive simplification steps (always taking braid images), then $\mathcal{K}_1$ and $\mathcal{K}_2$ are variants.

Proof. Part 1 is trivial, Part 3 follows immediately from Part 1, and Part 4 follows from Corollary 8.19 by a simple induction.

2. If $\mathcal{K}_1$ is irreducible and can be simplified to the braid $\mathcal{K}_2$, then this cannot be a proper simplification. By Lemma 4.10, $\mathcal{K}_1$ and $\mathcal{K}_2$ are variants. $\mathcal{K}_2$ cannot have a proper simplifier, since otherwise also $\mathcal{K}_1$ would have a proper simplifier, by Lemma 8.19. Hence the second statement holds.

5. Assume that $\mathcal{K}_1$ and $\mathcal{K}_2$ are two irreducible braids that can be reached from $\mathcal{K}$ by sequences of consecutive simplification steps, always taking braid images. Then there exist simplifiers (m_1, n_1) and (m_2, n_2) of $\mathcal{K}$ such that $\mathcal{K}_1 = \mathcal{K}^{\langle m_1,n_1\rangle}$ and $\mathcal{K}_2 = \mathcal{K}^{\langle m_2,n_2\rangle}$. The braid forests $\mathcal{K}^{(m_1,n_1)}$ and $\mathcal{K}^{(m_2,n_2)}$ are not necessarily irreducible. But we may add further simplification steps (m_1', n_1') and (m_2', n_2') such that $(\mathcal{K}^{(m_1,n_1)})^{(m_1',n_1')}$ and $(\mathcal{K}^{(m_2,n_2)})^{(m_2',n_2')}$ are irreducible. By Lemma 8.7, (m_1', n_1') and (m_2', n_2') are—obviously non-proper—simplifiers for $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. It follows that $\mathcal{K}_1$ and $\mathcal{K}_1^{\langle m_1',n_1'\rangle}$ are variants, and similarly for $\mathcal{K}_2$ and $\mathcal{K}_2^{\langle m_2',n_2'\rangle}$. By Theorem 8.15, the two braid forests $(\mathcal{K}^{(m_1,n_1)})^{(m_1',n_1')}$ and $(\mathcal{K}^{(m_2,n_2)})^{(m_2',n_2')}$ are variants. By Lemma 8.5, the two subbraids given by their roots—which are, by Lemma 4.20, $\mathcal{K}_1^{\langle m_1',n_1'\rangle}$ and $\mathcal{K}_2^{\langle m_2',n_2'\rangle}$—are variants. It follows that $\mathcal{K}_1$ and $\mathcal{K}_2$ are variants. □

8.3 Standard normalization

Let $\mathcal{O}_A^*$ be a subset of the set $\mathcal{O}_A$ of open atoms of A^Σ that has the same cardinality as the set of all "$\sim$"-equivalence classes of non-trivial [12] irreducible braids of type B. Similarly, let $\mathcal{O}_B^*$ be a subset of the set $\mathcal{O}_B$ of open atoms of B^Δ that has the same cardinality as the set of all "$\sim$"-equivalence classes of non-trivial irreducible braids of type A, let $\mathcal{O}^* := \mathcal{O}_A^* \cup \mathcal{O}_B^*$. We may now enumerate the elements of $\mathcal{O}^*$ in the form

$$\mathcal{O}_A^* = \{o_{[\mathcal{K}]} \mid \mathcal{K} \text{ is a nontrivial irreducible braid of type } B\},$$
$$\mathcal{O}_B^* = \{o_{[\mathcal{K}]} \mid \mathcal{K} \text{ is a nontrivial irreducible braid of type } A\}.$$

This means that $[\mathcal{K}] \mapsto o_{[\mathcal{K}]}$ establishes a bijection between the set of all "$\sim$"-equivalence classes of non-trivial irreducible braids of type A (B) and $\mathcal{O}_B^*$ ($\mathcal{O}_A^*$). In the sequel it is useful to imagine that each atom $o_{[\mathcal{K}]} \in \mathcal{O}_A^* \cup \mathcal{O}_B^*$ "stands for" the irreducible braid $\mathcal{K}$, modulo variants. We now define the standard normal form of a braid (forest) $\mathcal{K}$. Intuitively, a braid $\mathcal{K}$ with fibering function π is in standard normal form if each open atom o of $\mathcal{K}$ is in $\mathcal{O}_A^* \cup \mathcal{O}_B^*$ and stands for the subbraid of $\mathcal{K}$ with root $\pi(o)$, in the above sense. For the formal definition it is convenient to use the following notation. To each $o \in \mathcal{O}(\mathcal{K})$ we associate the unique subbraid $\mathcal{K}_o$ of $\mathcal{K}$ with root $\pi(o)$. Let $[\mathcal{K}_o^r]$ denote the "$\sim$"-equivalence class of an irreducible braid $\mathcal{K}_o^r$ that is obtained from $\mathcal{K}_o$ via

[12] compare Definition 4.4.

simplification. We have seen (Theorem 4.12) that the classes $[\mathcal{K}_o^r]$ coincide, for all possible choices of $\mathcal{K}_o^r$ and fixed o.

Definition 8.20 A braid (forest) $\mathcal{K}$ with fibering function π is in *standard normal form* if $\mathcal{O}(\mathcal{K}) \subseteq \mathcal{O}^*$ and if every open atom $o \in \mathcal{O}(\mathcal{K})$ always has the form $o = o_{[\mathcal{K}_o^r]}$.

Note that trivial braids are always in standard normal form.

Theorem 8.21 *Every braid (forest) in standard normal form is irreducible.*

Proof. It obviously suffices to consider braid forests. Let $\mathcal{K} = \langle a, C, D, \pi_A, \pi_B \rangle$ be a braid forest in standard normal form. Assume, to get a contradiction, that (m, n) is a proper simplifier for $\mathcal{K}$. Without loss of generality we may assume that $m(o_1) = m(o_2)$ for distinct open atoms $o_1, o_2 \in \mathcal{O}_A(C)$. Let $d_i := \pi_A(o_i)$, and let $\mathcal{K}_i$ denote the subbraid of $\mathcal{K}$ with root d_i, for $i = 1, 2$. Since (m, n) is a simplifier, $n(d_1) = n(d_2)$. By Lemma 8.7, (m, n) is a simplifier for $\mathcal{K}_1$ and $\mathcal{K}_2$. By Lemma 4.20, $\mathcal{K}_1^{\langle m,n \rangle}$ is the unique subbraid of $\mathcal{K}^{(m,n)}$ with root $n(d_i)$, for $i = 1, 2$. Since $n(d_1) = n(d_2)$, also $\mathcal{K}_1^{\langle m,n \rangle} = \mathcal{K}_2^{\langle m,n \rangle}$ which implies that $\mathcal{K}_1$ and $\mathcal{K}_2$ are variants. Since $\mathcal{K}$ is in standard normal form it follows that $o_1 = o_{[\mathcal{K}_1]} = o_{[\mathcal{K}_2]} = o_2$, which contradicts our assumption. □

Theorem 8.22 *Every braid (forest) $\mathcal{K}$ can be simplified to a unique braid (forest) in standard normal form.*

Proof. 1. Existence. We may assume that $\mathcal{K}$ is a braid forest, with fibering function π, say. Let m (n) denote the admissible endomorphism that maps each $o \in \mathcal{O}_A(\mathcal{K})$ $(o \in \mathcal{O}_B(\mathcal{K}))$ to $o_{[\mathcal{K}_o]}$ where $\mathcal{K}_o$ is an irreducible braid that is obtained via simplification of the subbraid of $\mathcal{K}$ with root $\pi(o)$. By Theorem 4.12, Part 5, m (n) is well-defined.

By Theorem 8.15 there exists a simplifier (m_1, n_1) of $\mathcal{K} = \langle a, C, D, \pi_A, \pi_B \rangle$ such that $\mathcal{K}_1 := \mathcal{K}^{(m_1,n_1)}$ is irreducible. If m_1 identifies the open atoms $o, o' \in \mathcal{O}_A(C)$, then n_1 identifies $d := \pi_A(o)$ and $d' := \pi_A(o')$. It follows that the two subbraids of $\mathcal{K}$ with roots $\pi(o)$ and $\pi(o')$ respectively both receive the same braid image under the simplification (m_1, n_1). This implies that $m(o) = m(o')$. It follows that the mapping $m_2 : m_1(o) \mapsto m(o)$ $(o \in \mathcal{O}_A(C))$ is well-defined. Symmetrically it follows that the mapping $n_2 : n_1(o) \mapsto n(o)$ $(o \in \mathcal{O}_B(D))$ is well-defined. Both mappings can be extended to admissible endomorphisms for which we shall use the same symbol.

Obviously $m_1 \circ m_2$ (resp. $n_1 \circ n_2$) and m (resp. n) coincide on $\mathcal{O}_A(C)$ (resp. $\mathcal{O}_B(D)$). Hence $m_1 \circ m_2$ (resp. $n_1 \circ n_2$) and m (resp. n) coincide on C (resp. D), by Lemma 8.1.

We now show that (m, n) is a simplifier for $\mathcal{K}$. Assume that $m(o) = m(o')$, for $o, o' \in \mathcal{O}_A(C)$. This means, by the definition of m, that all irreducible braid

images of the subbraids $\mathcal{K}'$ and $\mathcal{K}''$ of $\mathcal{K}$ with roots $d := \pi_A(o)$ and $d' := \pi_A(o')$ are variants. Let $\mathcal{K}'_1$ and $\mathcal{K}''_1$ denote the braid images of $\mathcal{K}'$ and $\mathcal{K}''$ under the simplification (m_1, n_1) respectively. $\mathcal{K}'_1$ and $\mathcal{K}''_1$ are subbraids of the irreducible braid forest $\mathcal{K}_1$, by Lemma 4.20. Part (b) of Lemma 8.16 shows that $\mathcal{K}'_1$ and $\mathcal{K}''_1$ are irreducible. This shows that $\mathcal{K}'_1$ and $\mathcal{K}''_1$ are variants. Now Part (c) of Lemma 8.16 shows that $\mathcal{K}'_1 = \mathcal{K}''_1$. The root of $\mathcal{K}'_1$ is $n_1(d)$, and the root of $\mathcal{K}''_1$ is $n_1(d')$. Hence $n_1(d) = n_1(d')$ and $n(d) = n_2(n_1(d)) = n_2(n_1(d')) = n(d')$.

Symmetrically it follows that $n(o) = n(o')$ always implies that $m(\pi_B(o)) = m(\pi_B(o'))$, for all $o, o' \in \mathcal{O}_B(D)$. This shows that (m, n) is in fact a simplifier for $\mathcal{K}$. Using Lemma 4.20 the definition of m and n shows that $\mathcal{K}^{(m,n)}$ is in standard normal form.

2. Uniqueness. We consider braid forests, the proof for braids is essentially the same. Assume that the braid forest $\mathcal{K}$ can be simplified to the braid forests $\mathcal{K}_1$ and $\mathcal{K}_2$ in standard normal form. The first part of the lemma shows that $\mathcal{K}_1$ and $\mathcal{K}_2$ are irreducible. By Theorem 8.15, $\mathcal{K}_1$ and $\mathcal{K}_2$ are variants. Hence there exists an admissible pair of automorphisms (m, n) such that $\mathcal{K}_2 = \mathcal{K}_1^{(m,n)}$. Let $o \in \mathcal{O}_A(\mathcal{K}_1)$, for $i = 1, 2$ let π_1 be the fibering function of $\mathcal{K}_i$, and let $\mathcal{K}'_1$ be the subbraid of $\mathcal{K}_1$ with root $\pi_1(o)$. Then o has the form $o_{[\mathcal{K}'_1]}$. On the other hand, $m(o) \in \mathcal{O}_A(\mathcal{K}_2)$ and the subbraid $\mathcal{K}'_2$ of $\mathcal{K}_2$ with root $\pi_2(m(o))$ is a variant of $\mathcal{K}'_1$. Since $\mathcal{K}_2$ is in standard normal form we obtain $m(o) = o_{[\mathcal{K}'_2]} = o_{[\mathcal{K}'_1]} = o$. We have seen that m is the Identity on $\mathcal{O}_A(\mathcal{K}_1)$. Similarly it follows that n is the Identity on $\mathcal{O}_B(\mathcal{K}_1)$. This shows that $\mathcal{K}_1 = \mathcal{K}_2$. □

We write $\mathcal{K}_\downarrow$ for the standard normal form of the braid $\mathcal{K}$. On the basis of Theorem 8.22 we may now write each element $o_{[\mathcal{K}]}$ of $\mathcal{O}^*$ in the form $o_{\mathcal{K}_\downarrow}$. This means that $\mathcal{K} \mapsto o_{\mathcal{K}}$ establishes a bijection between the set of all non-trivial braids in standard normal form of type A (B) and $\mathcal{O}_B^*$ ($\mathcal{O}_A^*$). □

Lemma 8.23 *A braid (forest) $\mathcal{K}$ with fibering function π is in standard normal form if and only if $\mathcal{O}(\mathcal{K}) \subseteq \mathcal{O}^*$ and if for every pair $\langle o_{\mathcal{K}'}, e\rangle \in \pi$ always $\mathcal{K}'$ is the unique subbraid of $\mathcal{K}$ with root e.*

Proof. Let $\mathcal{K}$ be in standard normal form. Obviously this implies that every subbraid of $\mathcal{K}$ is in standard normal form, too. Let $o \in \mathcal{O}(\mathcal{K})$, and let $\mathcal{K}_o$ be the subbraid of $\mathcal{K}$ with root $e = \pi(o)$. Then o has the form $o = o_{[\mathcal{K}_o^r]}$ where $\mathcal{K}_o^r$ is an irreducible braid obtained from $\mathcal{K}_o$ via simplification. Since $\mathcal{K}_o$ is irreducible, $o = o_{[\mathcal{K}_o]}$, which implies that o has the form $o = o_{\mathcal{K}_o}$. The converse direction is trivial. □

Lemma 4.21 *Let $\mathcal{K}$ be a braid (forest) with fibering function π. Let (m, n) denote the admissible pair of endomorphisms that maps each $o \in \mathcal{O}_A(\mathcal{K}) \cup \mathcal{O}_B(\mathcal{K})$ to $o_{\mathcal{K}'_\downarrow}$ where $\mathcal{K}'$ is the unique subbraid of $\mathcal{K}$ with root $\pi(o)$. Then (m, n) is a simplifier for $\mathcal{K}$ and each subbraid of $\mathcal{K}^{(m,n)}$ is in standard normal*

form.

Proof. Since for $o \in \mathcal{O}_A^* \cup \mathcal{O}_B^*$ always $o_{[\mathcal{K}]}$ and $o_{\mathcal{K}_\downarrow}$ denote the same atom, the proof of the existence statement of the previous theorem shows that (m, n) is a simplifier for $\mathcal{K}$ and that $\mathcal{K}^{(m,n)}$ is in standard normal form. This obviously implies that each subbraid of $\mathcal{K}^{(m,n)}$ is in standard normal form. □

Lemma 8.24 *Given $e \in (A_* \cup B_*) \setminus (\mathcal{O}_A^* \cup \mathcal{O}_B^*)$ there exists a unique braid $\mathcal{K} \in A \odot B$ such that e is the root of $\mathcal{K}$.*

Proof. Existence. Let $e \in (A_* \cup B_*) \setminus (\mathcal{O}_A^* \cup \mathcal{O}_B^*)$. We may assume that $e \in A_* \setminus \mathcal{O}_A^*$. Let $\mathcal{O}_A(e) = \{o_1, \ldots, o_n\}$, and let $\mathcal{K}_i = \langle e_i, C_i, D_i, \pi_A^i, \pi_B^i \rangle$ be the braid in standard normal form such that $o_i = o_{K_i}$. Let $C := \bigcup_{i=1}^n C_i \cup \{e\}$, $D := \bigcup_{i=1}^n D_i$, $\pi_A := \bigcup_{i=1}^n \pi_A^i \cup \{\langle o_{\mathcal{K}_i}, e_i \rangle \mid i = 1, \ldots, n\}$, and $\pi_B := \bigcup_{i=1}^n \pi_B^i$. Then $\mathcal{K} = \langle e, C, D, \pi_A, \pi_B \rangle \in A \odot B$ has root e.

Uniqueness. Let $\mathcal{K} = \langle e, C, D, \pi_A, \pi_B \rangle \in A \odot B$. Since, by Lemma 8.23, for each open atom $o = o_{K_i}$ in $\mathcal{O}_A(e)$ always $\mathcal{K}_i$ is the subbraid of $\mathcal{K}$ with root $\pi(o)$, the structure of $\mathcal{K}$ is obviously completely determined by e. □

9 Conclusion

In this paper we have introduced rational amalgamation, a general methodology for combining constraint systems. The present work, in connection with the discussion of free amalgamation in [4], seems to suggest a new view of the problem of combining solution domains and constraint solvers. There is now strong evidence that the situation considered in [4] and in this paper—the construction of "mixed" elements of a combined domain, given the "pure" elements of two component structures as construction units—is quite similar to the process of building the elements of a single structure, given the symbols of a fixed signature as construction units. We are confident that this analogy will help to isolate the most important methods for combining structures over disjoint signatures, and to understand the relationship and the differences between different amalgamation constructions.

When we compose elements, given the symbols of a fixed signature, three different structures may be obtained in a direct way, delost on the composition principle, namely the free term algebra, the algebra of rational trees, and the algebra of infinite trees. The privileged role of these three algebras, and the rich amount of interesting relationships between them, are now well-understood (e.g., [10,13]). We believe that free amalgamation, rational amalgamation and a further construction called "infinite amalgamation" (still to be investigated) reflect this role on the higher level of amalgamation constructions. Many of the results that we have obtained for free and rational amalgamation can be

interpreted in this sense:

- The universality-property of the free amalgamated product (see [4]) reflects the status of the free term algebra as the absolutely free Σ-algebra.
- We have seen that the free amalgamated product is always a substructure of the rational amalgamated product. This reflects the fact that the free term algebra is always a substructure of the algebra of rational trees.
- It is well-known that the unification algorithm for the algebra of rational trees can be considered as the variant of the unification algorithm for the free term algebra where we omit the occur-check. Similarly, the decomposition scheme for rational amalgamation as given here is—essentially—the decomposition scheme for free amalgamation where we omit the "interstructural" occur-check that is provided by the choice of a linear ordering in the latter scheme.

We would not be surprised if much more principles, techniques and theorems, well-known on the level of tree constructions, could be lifted to the level of combining structures. Our experience with rational amalgamation seems to indicate that this is a difficult, but promising line of research if we want to understand the scale of possibilities, and the limitations for combining solution domains and constraint solvers.

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