Negation in Combining Constraint Systems *

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Abstract. In a recent paper, Baader and Schulz presented a general method for the combination of constraint systems for purely positive constraints. But negation plays an important role in constraint solving. E.g., it is vital for constraint entailment. Therefore it is of interest to extend their results to the combination of constraint problems containing negative constraints. We show that the combined solution domain introduced by Baader and Schulz is a domain in which one can solve positive and negative "mixed" constraints by presenting an algorithm that reduces solvability of positive and negative "mixed" constraints to solvability of pure constraints in the components. The existential theory in the combined solution domain is decidable if solvability of literals with so-called linear constant restrictions is decidable in the components. We also give a criterion for ground solvability of mixed constraints in the combined solution domain. The handling of negative constraints can be significantly simplified if one can show that the solution domain owns the independence of negative constraints property. We provide a modularity result giving sufficient conditions for the component systems in order for the combined solution domain to have the independence property.

1 Introduction

One idea behind constraint solving is to use specialised formalisms and inference mechanisms to solve domain-specific tasks. In many applications, however, one is faced with a complex combination of different problems, which means that a system tailored to solving a single problem can only be applied if it is possible to combine it with other specialised systems. In a recent paper [3], Baader and Schulz present a general method for the combination of constraint systems. Their method is applicable to a large class of structures, the so called quasi-free structures. Quasi-free structures comprise many important infinite non-numerical domains such as term algebras and quotient term algebras, rational tree algebras, vector spaces, hereditarily finite well-founded and non-wellfounded lists, sets and multi sets as well as features structures of the Smolka & Treinen variety. The combined solution domain they present, the

*This work was supported by a DFG grant (SPP "Deduktion") and by the Esprit working group 22457 – CCL II of the EU.

so called free amalgamated product, is characterised by the property of being the most general combined solution domain in the sense that every other combined solution domain contains a homomorphic image of the free amalgamated product. The authors also give an algorithm to reduce solvability of "mixed" constraints, i.e., constraints over the joint signature, in the free amalgamated product to pure constraints in the components. With the help of this algorithm, they show that the positive theory of the free amalgamated product is decidable provided the positive theory of both components is decidable. But negation is not addressed in this paper. Indeed, the proof methods used are such that they cannot be extended to include negation.

Negation, however, plays an important role in constraint solving. For example, it is required for implication and constraint entailment, and both are used heavily in actual implementations of constraint solvers when reducing sets of constraints (see, e.g., [9]). Its usefullness for practical applications of constraint techniques is obvious. Hence it is the aim of this paper to investigate to what extend constraint systems with negation can be combined. We show how solvability of positive and negative mixed constraints in the free amalgamated product can be reduced to solvability of pure constraints in the components using an extension of the algorithm by Baader and Schulz. The existential theory of the free amalgamation is decidable provided solvability of conjunctions of literals with so called linear constant restrictions is decidable in the components. When solving negative constraints, one is often interested in the existence of ground solutions. Therefore we present sufficient conditions for solutions in the components to give a ground solution in the free amalgamation. The existence of a ground solution of mixed constraints in the free amalgamated product is decidable, if the existence of so-called restrictive solutions of pure constraints with linear constant restrictions is decidable in the components.

In the second half of this paper, we deal with the independence property of negative constraints. A constraint system is said to have the independence property, if the solvability of a conjunction of positive constraints and a conjunction of negative constraints can be reduced to solving each single negative conjunct together with the conjunction of positive constraints. In other words, if for some conjunction of negative constraints, each of them – plus the positive constraints – is solvable separately, then their conjunction is solvable. This property plays an important role in real world constraint solvers that cope with negative constraints (see, e.g., [9]).

The aim of this part is to give a modularity result for the independence property, that is to state under which conditions the free amalgamated product owns the independence property provided the component structures do so. We will finally show: The free amalgamation of two unitary regular and non-collapsing quasi-free structures has the independence property. To get there is quite a way. First we look at a particular subclass of quasi-free structures, namely equational theories, since there is not yet any discussion of this topic for these prototypical quasi-free structures. We find that unitary equational theories have the independence property, while finitary do not. And the combination of unitary regular and collapse-free equational theories is again unitary and hence has the independence property. We lift these results to quasi-free structures in general. Unitary quasi-free structures have the independence property. And the free amalgamation of two unitary regular and non-collapsing quasi-free structures is unitary, hence has the independence property.

Some of the results we present are inspired by [2], which describes combination techniques for equational disunification algorithms. For example, our decomposition algorithm (cf. 4.4) is a generalisation of the one given in [2]. But it should be noted that the proof methods of that paper – heavy use of rewriting techniques – are *not* extendable to the general, algebraic view that we take here.

For lack of space, we had to omit most proofs. They can be found in Chapter 6 of [10].

2 Preliminaries

A signature Σ consists of a set Σ_F of function symbols and a disjoint set Σ_P of predicate symbols (not containing "="), each of fixed arity. Expressions $\mathfrak{A}^\Sigma$ denote Σ-structures over the carrier set A, and $f_\mathfrak{A}$ ($p_\mathfrak{A}$) stands for the interpretation of $f \in \Sigma_F$ ($p \in \Sigma_P$) in $\mathfrak{A}^\Sigma$. Σ-terms $(t, t_1, \ldots)$ and atomic Σ-formulas (of the form $t_1 = t_2$, or of the form $p(t_1, \ldots, t_n)$) are built as usual. A Σ-formula φ is written in the form $\varphi(v_1, \ldots, v_n)$ in order to indicate that the set $Var(\varphi)$ of free variables of φ is a subset of $\{v_1, \ldots, v_n\}$. We write $\mathfrak{A}^\Sigma \models \varphi(v_1/a_1, \ldots, v_n/a_n)$ if φ becomes true in $\mathfrak{A}^\Sigma$ under all assignments that map v_i to $a_i \in A$, for $1 \leq i \leq n$.

Σ-homomorphisms, Σ-isomorphisms, and Σ-endomorphisms are defined as usual, see e.g., [13]. With $End_\mathfrak{A}^\Sigma$ we denote the monoid of all endomorphisms of $\mathfrak{A}^\Sigma$, with composition as operation. If $g : A \to B$ and $h : B \to C$ are mappings, then $h \circ g : A \to C$ denotes their composition. If M is a set, then $|M|$ denotes its cardinality.

The constraint problems we consider in this paper are conjunctions of literals. Clearly, every such constraint problem Γ can be written as $\Gamma^+ \wedge \bigwedge_{i=1}^k C_i^-$ where Γ^+ is the conjunction of all positive constraints (atoms) of Γ and $\bigwedge_{i=1}^k C_i^-$ is the conjunction of all negative constraints (negated atoms) of Γ. If we do not need to refer to specific negative constraints, we abbreviate $\bigwedge_{i=1}^k C_i^-$ by Γ^-.

Definition 2.1 A structure $\mathfrak{A}$ has the *independence of negative constraints* property, iff for every constraint problem Γ we have:

Γ is solvable in $\mathfrak{A}$, iff
for every $i = 1, \ldots, k$ the conjunction $\Gamma^+ \wedge C_i^-$ is solvable in $\mathfrak{A}$.

In this paper, a constraint problem is a conjunction of literals, i.e., of atomic and negated atomic formulae.

3 Quasi-free Structures and the Free Amalgamated Product

In this section we introduce the class of structures we use as solution domains for constraint solving. First we recall the definition of quasi-free structures given in [3]. We consider a fixed Σ-structure $\mathfrak{A}^\Sigma$.

Definition 3.1 Let A_0, A_1 be subsets of $\mathfrak{A}^\Sigma$. Then A_0 *stabilises* A_1 iff all elements m_1 and m_2 of $End_{\mathfrak{A}}^{\Sigma}$ that coincide on A_0 also coincide on A_1. For $A_0 \subseteq A$ the *stable hull* of A_0 is the set

$$\mathrm{SH}^{\mathfrak{A}}(A_0) := \{a \in A \mid A_0 \ \text{ stabilises } \{a\}\}.$$

$\mathrm{SH}^{\mathfrak{A}}(A_0)$ is always a Σ-substructure of $\mathfrak{A}^\Sigma$, and $A_0 \subseteq \mathrm{SH}^{\mathfrak{A}}(A_0)$. The stable hull of A_0 can be larger than the Σ-subalgebra generated by A_0.

Definition 3.2 The set $X \subseteq A$ is an *atom set* for $\mathfrak{A}^\Sigma$ if every mapping $X \to A$ can be extended to an endomorphism.

Definition 3.3 A countably infinite Σ-structure $\mathfrak{A}^\Sigma$ is a *quasi-free structure* iff $\mathfrak{A}^\Sigma$ has an infinite atom set X where every $a \in A$ is stabilised by a finite subset of X. We denote this quasi-free structure by $(\mathfrak{A}^\Sigma, X)$.

Examples 3.4 The class of quasi-free structures contains many important non-numerical infinite solution domains. For example, all free structures (see, e.g., [13]), rational tree algebras ([6, 12]), feature structures with arity ([17, 5]), domains with nested, finite or rational lists (rational lists are used in Prolog III, see [7]), and domains with nested, finite or rational sets (as introduced in [1] and used in [14]). In each case, we have to take the non-ground variant since we assume the existence of a countably infinite set of atoms. For details we refer to [3].

A fundamental property of quasi-free structures is the following (see [3], Lemma 3.18): for each $a \in A$ there exists a *unique minimal* finite set $Y \subseteq X$ such that $a \in \mathrm{SH}^{\mathfrak{A}}(Y)$.

Definition 3.5 The *stabiliser* of $a \in A$, $\mathrm{Stab}^{\mathfrak{A}}(a)$, is the unique minimal finite subset Y of X such that $a \in \mathrm{SH}^{\mathfrak{A}}(Y)$. The stabiliser of $A' \subseteq A$ is the set $\mathrm{Stab}^{\mathfrak{A}}(A') := \bigcup_{a \in A'} \mathrm{Stab}^{\mathfrak{A}}(a)$.

Let $(\mathfrak{A}^\Sigma, X)$ be a quasi-free structure. $(\mathfrak{A}^\Sigma, X)$ is called *quasi-free for a structure* $\mathfrak{D}^\Sigma$, iff every mapping $X \to D$ can be extended to a homomorphism $\mathfrak{A}^\Sigma \to \mathfrak{D}^\Sigma$.

Definition 3.6 Let $(\mathfrak{A}^\Sigma, X)$ and $(\mathfrak{B}^\Delta, X)$ be two quasi-free structures. The structure $\mathfrak{D}^{\Sigma \cup \Delta}$ is an *amalgamation* of $\mathfrak{A}^\Sigma$ and $\mathfrak{B}^\Delta$, iff there exist homomorphisms $d_A : \mathfrak{A}^\Sigma \to \mathfrak{D}$ and $d_B : \mathfrak{B}^\Delta \to \mathfrak{D}$ such that $d_A(x) = d_B(x)$ for all $x \in X$.

Definition 3.7 Let $(\mathfrak{A}^\Sigma, X)$ and $(\mathfrak{B}^\Delta, X)$ be two quasi-free structures. The amalgamation $(\mathfrak{A}^\Sigma \otimes \mathfrak{B}^\Delta, e_A, e_B)$ is the *free amalgamated product*, iff for every amalgamation $(\mathfrak{D}^{\Sigma\cup\Delta}, d_A, d_B)$ such that $(\mathfrak{A}^\Sigma, X)$ is quasi-free for $\mathfrak{D}^\Sigma$ and $(\mathfrak{B}^\Delta, X)$ is quasi-free for $\mathfrak{D}^\Delta$ there exists a unique homomorphism $h! : \mathfrak{A}^\Sigma \otimes \mathfrak{B}^\Delta \to \mathfrak{D}^{\Sigma\cup\Delta}$ such that $h! \circ e_A = d_A$ and $h! \circ e_B = d_B$.

$$
\begin{tikzcd}
 & \mathfrak{A}^\Sigma \arrow[dr, "e_A"] \arrow[drr, "d_A"] & & \\
X \arrow[ur, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & & \mathfrak{A}^\Sigma \otimes \mathfrak{B}^\Delta \arrow[r, "h!"] & \mathfrak{D}^{\Sigma\cup\Delta} \\
 & \mathfrak{B}^\Delta \arrow[ur, "e_B"] \arrow[urr, "d_B"'] & &
\end{tikzcd}
$$

In [3], Baader and Schulz provide a construction method to obtain the free amalgamated product of two arbitrary quasi-free structures.

4 Free Amalgamation of Negative Constraints: The General Case

In this section, we show that solving mixed constraints in the free amalgamation can be reduced to solving pure constraints in the components, even if the constraints contain negations. The type of problems we have to solve in the components are constraint problems with generalised linear constant restrictions. Problems with linear constant restrictions are defined in [2].

Definition 4.1 Let Γ be a constraint problem of signature Σ and Δ be some disjoint signature. A *generalised linear constant restriction* $L = (\Pi, Lab, <_L)$ consists of a a partition Π of $Var(\Gamma)$, a labelling function $Lab : Var(\Gamma) \to \{\Sigma, \Delta\}$ and a linear order $<_L$ on $Var(\Gamma)$ such that for all $x, x', y, y' \in Var(\Gamma)$: if $x \equiv_\Pi x'$[1], $y \equiv_\Pi y'$ then $Lab(x) = Lab(x')$ and $x <_L y$ implies $x' <_L y'$.

An assignment σ solves (Γ, L) in $(\mathfrak{A}^\Sigma, X)$, iff it solves Γ and
for each $x, y \in Var(\Gamma) : \sigma(x) = \sigma(y)$ iff $x \equiv_\Pi y$,
for each $x \in Var(\Gamma)$ with $Lab(x) = \Delta : \sigma(x) \in X$, and
for each $x, y \in Var(\Gamma)$ with $Lab(x) = \Delta, Lab(y) = \Sigma, y <_L x$: $\sigma(x) \notin \mathrm{Stab}^{\mathfrak{A}}(\sigma(y))$.

Now we can present the main theorems of the first part.

Theorem 4.2 *Let $(\mathfrak{A}^\Sigma, X)$ and $(\mathfrak{B}^\Delta, X)$ be two quasi-free structures over disjoint signatures. The solvability of mixed $\Sigma \cup \Delta$ constraint problems, i.e., conjunctions of literals, in the free amalgamated product $\mathfrak{A}^\Sigma \otimes \mathfrak{B}^\Delta$ is decidable, if solvability of constraint problems with generalised linear constant restrictions is decidable in both components $(\mathfrak{A}^\Sigma, X)$ and $(\mathfrak{B}^\Delta, X)$.*

[1] $x \equiv_\Pi x'$ means x and x' belong to the same equivalence class of Π.

Since existential quantifiers distribute over disjunctions, we can strengthen our result a little bit.

Theorem 4.3 *Let $(\mathfrak{A}^\Sigma, X)$ and $(\mathfrak{B}^\Delta, X)$ be two quasi-free structures over disjoint signatures. The existential theory of the free amalgamated product $\mathfrak{A}^\Sigma \otimes \mathfrak{B}^\Delta$ is decidable, if solvability of constraint problems with generalised linear constant restrictions is decidable in both components $(\mathfrak{A}^\Sigma, X)$ and $(\mathfrak{B}^\Delta, X)$.*

Decomposition Algorithm

In order to show the above theorems, we present a decomposition algorithm that reduces solvability of mixed constraints in the free amalgamated product to solvability of pure constraints with generalised linear constant restrictions in the components. The algorithm is a simple generalisation of the one given in [2]. We assume that the input problem Γ is in *decomposed form*, i.e., $\Gamma = \Gamma_\Sigma \wedge \Gamma_\Delta \wedge \Gamma_{\neq}$ where Γ_Σ (resp. Γ_Δ) contains only pure Σ- (resp. Δ-) literals and no disequations; and $\Gamma_{\neq}$ contains only disequations between variables. Every input problem can be transformed into an equivalent one in decomposed form by means of a polynomial algorithm which is now standard in the field (see, e.g., [2]). A *shared* variable is a variable occurring both in Γ_Σ and Γ_Δ.

Algorithm 4.4 The input Γ is a constraint problem over signature $\Sigma \cup \Delta$ in decomposed form.

Step 1: variable identification
Non-deterministically choose a partition Π of the set of all shared variables of the constraint problem such that whenever the constraint problem contains a disequation $x \neq y$ then x and y belong to different classes of the partition.

Step 2: variable labelling and ordering
For a given system, choose a mapping Lab from the set of shared variables into the set of theory labels $\{\Sigma, \Delta\}$ and a strict linear order $<_L$ on the shared variables such that both respect the partition Π. This triple $L = (\Pi, Lab, <_L)$ gives raise to a generalised linear constant restriction.

Step 3: split systems
A given system Γ is split into two systems $\Gamma_{3,\Sigma} \wedge \Gamma_{3,\Delta}$ where $\Gamma_{3,\Sigma} = \Gamma_\Sigma \wedge \Gamma_{\neq}$ and $\Gamma_{3,\Delta} = \Gamma_\Delta \wedge \Gamma_{\neq}$. The subsystems can now be considered as constraint problems with generalised linear constant restriction.

Proposition 4.5 *The input problem Γ is solvable, if and only if there exists an output pair $(\Gamma_{3,\Sigma}, \Gamma_{3,\Delta})$ and a generalised linear constant restriction L such that $(\Gamma_{3,\Sigma}, L)$ is solvable in $\mathfrak{A}^\Sigma$ and $(\Gamma_{3,\Delta}, L)$ is solvable in $\mathfrak{B}^\Delta$.*

The difficult part of this proposition is to show that the existence of solutions for $(\Gamma_{3,\Sigma}, L)$ and $(\Gamma_{3,\Delta}, L)$ implies the existence of a solution for Γ. We will sketch the proof here. To do so, we have to present how the free amalgamated product

of two quasi-free structures $(\mathfrak{A}^{\Sigma}, X)$ and $(\mathfrak{B}^{\Delta}, X)$ is constructed. The full details of the construction can be found in [3]. We embed both into isomorphic super structures $(\mathfrak{A}^{\Sigma}_{\infty}, Y_1)$ and $(\mathfrak{B}^{\Delta}_{\infty}, Y_2)$ such that $Y_i \setminus X$ is infinite for $i = 1, 2$. These structures are fibred in the following way. A *fibre* is either of the form $\{x\}$ with $x \in X$ or $\{b, y\}$ where $y \in Y_1 \setminus X$ and $b \in B_{\infty} \setminus Y_2$ or $y \in Y_2 \setminus X$ and $b \in A_{\infty} \setminus Y_1$. Let $d_1, d_2, d_3, d_4, \ldots$ be some enumeration of $A_{\infty} \cup B_{\infty}$. We define an ascending tower of fibre sets $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$ where each $\mathcal{F}_i$ is a set of mutually disjoint fibres. We start with $\mathcal{F}_0 := \{\{x\} \mid x \in X\}$. Suppose $\mathcal{F}_k$ is already defined. To define $\mathcal{F}_{k+1}$ we distinguish two cases.
Case 1: There exists an element $d \in A_{\infty} \cup B_{\infty}$, say in A_{∞}, such that each element of the stabiliser $\mathrm{Stab}^{\mathfrak{A}_{\infty}}_{\Sigma}(d)$ belongs to a fibre in $\mathcal{F}_k$, but d itself does not. Then we proceed as follows: Let d_{min} be the smallest such element (in the enumeration) and, say, $d_{min} \in A_{\infty}$. We select an atom $y \in Y_2 \setminus X$ which does not belong to any fibre in $\mathcal{F}_k$ and define $\mathcal{F}_{k+1} := \mathcal{F}_k \cup \{\{d_{min}, y\}\}$.
Case 2: Otherwise define $\mathcal{F}_{k+1} := \mathcal{F}$.

Let $\mathcal{F} := \bigcup_{k \geq 0} \mathcal{F}_k$. Then $D := \bigcup_{F \in \mathcal{F}} F$ is a subset of $A_{\infty} \cup B_{\infty}$. Let $A_{\diamond} := D \cap A_{\infty}$, $B_{\diamond} := D \cap B_{\infty}$ and $Z_i = D \cap Y_i$. $(\mathfrak{A}^{\Sigma}_{\diamond}, Z_1)$ (resp. $(\mathfrak{B}^{\Delta}_{\diamond}, Z_2)$) is an isomorphic substructure of $(\mathfrak{A}^{\Sigma}_{\infty}, Y_1)$ (resp. $(\mathfrak{B}^{\Delta}_{\infty}, Y_2)$). Since each element $d \in D$ belongs to a unique fibre F_d of $\mathcal{F}$, the fibring gives rise to a bijection $h : A_{\diamond} \to B_{\diamond}$ that is used to superimpose the Σ-structure of $\mathfrak{A}^{\Sigma}_{\diamond}$ onto $B_{\diamond}$ and the Δ-structure of $\mathfrak{B}^{\Delta}_{\diamond}$ onto $A_{\diamond}$. Thus h can be seen as a $\Sigma \cup \Delta$-isomorphism $\mathfrak{A}^{\Sigma \cup \Delta}_{\diamond} \to \mathfrak{B}^{\Sigma \cup \Delta}_{\diamond}$ and $\mathfrak{A}^{\Sigma} \otimes \mathfrak{B}^{\Delta} := \mathfrak{A}^{\Sigma \cup \Delta}_{\diamond}$.

The *shadow* of an element $a \in D$ is recursively defined as $\mathrm{Sd}(a) := \{a\} \cup \mathrm{Stab}(a) \cup \bigcup \{\mathrm{Sd}(h(x)) \mid x \in \mathrm{Stab}(a)\}$, so it contains the element a, its stabiliser, the fibre images of the stabiliser elements, their stabiliser, the fibre images thereof and so on, everything pending below a via stabilisers and fibre images. Due to the inductive definition of the fibring construction, the shadow of each element is finite.

Now to the soundness proof. Suppose there is a solution σ_{Σ} of $(\Gamma_{3,\Sigma}, L)$ in $\mathfrak{A}^{\Sigma}$ and a solution σ_{Δ} of $(\Gamma_{3,\Delta}, L)$ in $\mathfrak{B}^{\Delta}$. Then there is a solution σ'_{Σ} of $(\Gamma_{3,\Sigma}, L)$ in $\mathfrak{A}^{\Sigma}_{\diamond}$ and a solution σ'_{Δ} of $(\Gamma_{3,\Delta}, L)$ in $\mathfrak{B}^{\Delta}_{\diamond}$. If we had $h(\sigma'_{\Sigma}(v)) = \sigma'_{\Delta}(v)$ for all $v \in \mathit{Var}(\Gamma)$, then we would be done, because then, obviously, σ'_{Σ} would be a solution of $\Gamma_{3,\Delta}$ in $\mathfrak{A}^{\Sigma \cup \Delta}_{\diamond}$. So the task is to show that the given solutions σ'_{Σ} and σ'_{Δ} can be transformed by means of automorphisms in such a way that finally the value of a variable $v \in \mathit{Var}(\Gamma)$ under σ'_{Σ} is the fibre image of the value under σ'_{Δ}. We call this the fibring condition for v.

The generalised linear constant restriction L contains a linear order $<_L$ of the variables in Γ. Let $v_1, v_2, v_3, \ldots, v_n$ be the enumeration of the variables alongside the order. By induction on this enumeration we prove that there exist solutions $l_{i,\Sigma}$ of $(\Gamma_{3,\Sigma}, L)$ and $l_{i,\Delta}$ of $(\Gamma_{3,\Delta}, L)$ such that for all variables v_j with $j \leq i$ we have $h \circ l_{i,\Sigma}(v_j) = l_{i,\Delta}(v_j)$, that is the first i variables fulfil the fibring condition. But this simple statement alone is too weak. In an induction step, we may need to apply an automorphism exchanging two atoms in order to establish the fibring condition for the current variable. The difficult part consists in showing that this automorphism does not dissolve the fibring conditions for variables already handled. That is where the shadows of elements come in. We assume that every variable occurring in the shadows of the first i variables already fulfils the fibring condition. And we show that the automorphisms are the identity on the shadows of the first i variables and hence do not dissolve an already established fibring condition.

We first look at a simple case. Let $V := \mathit{Var}(\Gamma)$ and $n := |V|$. We partition V as follows. Define $V_Z := \{v \in V \mid \sigma'_\Sigma(v) \in Z_1 \text{ and } \sigma'_\Delta(v) \in Z_2\}$ as the set of all variables that are mapped to atoms in both solutions, and let $V_{\bar{Z}} := V \setminus V_Z$ be its complement. Let X be partitioned as $X_0 \uplus X_1 \uplus X_2$ such that X_1 and X_2 are infinite and X_0 has at least n elements. For a variable $v \in V_Z$, we pick some atom $z \in X_0$, the common atom set, and apply the automorphism that transposes z and $\sigma'_\Sigma(v)$ and the automorphism that transposes z and $\sigma'_\Delta(v)$. Iterating this process with other variables if necessary, we can ensure that if both solutions map a variable to an atom, this atom is one and the same in the common atom set X_0, and thus the fibring condition is fulfilled. These are our new solutions σ''_Σ and σ''_Δ.

Now, we prove by induction the following properties for each $i \leq n$:

There are solutions $l_{i,\Sigma}$ of $(\Gamma_{3,\Sigma}, L)$ in $\mathfrak{A}_\diamond^\Sigma$ and $l_{i,\Delta}$ of $(\Gamma_{3,\Delta}, L)$ in $\mathfrak{B}_\diamond^\Delta$ such that

1. For each variable $v \in V_Z$ holds $h \circ l_{i,\Sigma}(v) = l_{i,\Delta}(v) \in X_0$. We cared of these variables before. Now we have to ensure that this property is not lost.

2. Let $S_i := \bigcup_{j \leq i} \mathrm{Sd}(l_{i,\Sigma}(v_j)) \cup \bigcup_{j \leq i} \mathrm{Sd}(l_{i,\Delta}(v_j))$ be the union of all shadows of all solutions to variables up to v_i in the enumeration. For each variable v such that $l_{i,\Sigma}(v) \in S_i$ or $l_{i,\Delta}(v) \in S_i$ holds $h \circ l_{i,\Sigma}(v) = l_{i,\Delta}(v)$. So, we do not just demand that for all variables in the enumeration up to v_i the fibring condition holds, we also demand that for all elements in their shadows that are solutions to some variables (maybe occurring only later in the enumeration).

3. $\mathrm{Stab}(l_{i,\Sigma}(V_{\bar{Z}})) \subseteq S_i \cup X_1, \mathrm{Stab}(l_{i,\Delta}(V_{\bar{Z}})) \subseteq S_i \cup X_2$. This is a technical condition on the stabilisers of solution elements to variables where one of the two solutions is a non-atom. It gives a strong control over the stabilisers of non-atom solutions. Either they are in S_i, which means we have already taken care of them. Or they are in X_1 (resp. X_2), which means they are harmless.

Proving these properties for the base case provides no difficulties. The induction step is done by a complicated and technical argument by cases. We have two simple cases: The next variables $v_{i+1} \in V_Z$ the set of all variables mapped to atoms in both solutions or $v_{i+1} \in V_{\bar{Z}}$ and $h \circ l_{i,\Sigma}(v_{i+1}) = l_{i,\Delta}(v_{i+1})$. In both cases we define $l_{i+1,\Sigma} := l_{i,\Sigma}$ and $l_{i+1,\Delta} := l_{i,\Delta}$ and can check with not to big an effort that Properties 1 to 3 hold for $i+1$.

The important and interesting case is where $v_{i+1} \in V_{\bar{Z}}$ and $h \circ l_{i,\Sigma}(v_{i+1}) \neq l_{i,\Delta}(v_{i+1})$. We assume that $\mathit{Lab}(v_{i+1}) = \Sigma$, which implies that $y_{i+1} := l_{i,\Delta}(v_{i+1}) \in Z_2$, and $z_{i+1} := h \circ l_{i,\Delta}(v_{i+1}) \in Z_2$ by definition of the fibres. The transposition that exchanges y_{i+1} and z_{i+1} can be extended to a unique automorphism τ_{i+1} of $\mathfrak{B}_\diamond^\Delta$. Thus we define $l_{i+1,\Delta} := \tau_{i+1} \circ l_{i,\Delta}$ and set $l_{i+1,\Sigma} := l_{i,\Sigma}$. Then, clearly, $l_{i+1,\Sigma}$ is a solution of $(\Gamma_{3,\Sigma}, L)$ and $l_{i+1,\Delta}$ one of $(\Gamma_{3,\Delta}, L)$. And

$h \circ l_{i+1,\Sigma}(v_{i+1}) = l_{i+1,\Delta}(v_{i+1})$ by definition of $l_{i+1,\Delta}$. Showing that the Properties 1 to 3 still hold is technical. It is based on the following observations.

(a) $z_{i+1} \in Z_2 \setminus X$ and especially $z_{i+1} \notin X_2$ or X_1. This is a consequence of the fibring constructions.

(b) $z_{i+1} \notin S_i$. If z_{i+1} were in S_i, then $z_{i+1} = y_{i+1}$ by induction hypothesis.

(c) $z_{i+1} \notin \mathrm{Stab}(l_{i,\Sigma}(V_{\bar{Z}}))$ and $z_{i+1} \notin \mathrm{Stab}(l_{i,\Delta}(V_{\bar{Z}}))$, because $\mathrm{Stab}(l_{i,\Sigma}(V_{\bar{Z}})) \subset S_i \cup X_1$ and $\mathrm{Stab}(l_{i,\Delta}(V_{\bar{Z}})) \subset S_i \cup X_2$.

(d) $y_{i+1} \in X_2, y_{i+1} \notin S_i$. $y_{i+1} \in S_i \cup X_2$ by induction hypothesis and $y_{i+1} \in S_i$ would imply $y_{i+1} = z_{i+1}$.

It is important to see, what τ_{i+1} does. By (a) and (d), τ_{i+1} is the identity on X_0. Therefore for all $v \in V_Z : l_{i+1,\Delta}(v) = \tau_{i+1} \circ l_{i,\Delta}(v) = l_{i,\Delta}(v) = h \circ l_{i+1,\Sigma}(v) \in X_0$. This is Property 1.

By (c) and (d) and the above, τ_{i+1} is the identity on $\mathrm{Stab}(l_{i+1,\Sigma}(V))$ and therefore also on $l_{i+1,\Sigma}(V)$. That means τ_{i+1} does not touch the solution $l_{i+1,\Sigma}$. By (b) and (d), τ_{i+1} is also the identity on S_i. This is very important, since it states that all the nice properties of S_i are preserved. This insight together with the fact that $S_{i+1} = S_i \cup \mathrm{Sd}(l_{i+1,\Delta}(v_{i+1}))$ can then be used to show Properties 2 and 3 the techniqualities of which we will spare you.

For $i = n$ this induction shows there exists a solution $l_{n,\Sigma}$ of $\Gamma_{3,\Sigma}$ and a solution $l_{n,\Delta}$ of $\Gamma_{3,\Delta}$. Furthermore $l_{n,\Delta} = h \circ l_{n,\Sigma}$, hence $l_{n,\Sigma}$ is also a solution of $\Gamma_{3,\Delta}$ in $\mathfrak{A}_{\diamond}^{\Sigma\cup\Delta} = \mathfrak{A}^{\Sigma} \otimes \mathfrak{B}^{\Delta}$. Therefore there exists a solution of Γ.

A Stronger Combination Result?

We showed that the *existential* fragment of the free amalgamated product is decidable provided conjunctions of literals with linear constant restrictions are decidable in the components. A natural question to ask is, if we can decide a larger quantifier prefix fragment than just the existential one. The answer we find is a negative one. We will provide counterexamples from the field of equational unification.

Solvability of disunification problems with linear constant restrictions for the free theory and the theory of an AC-function symbol is shown to be decidable by F. Baader and K. U. Schulz in [2]. Therefore the existential theory of the free algebra of one AC-symbol and any finite number of free function symbols is decidable. Even better, the full first order theory for both the free theory [12] and the theory of an AC-function symbol and any number of constants [16] is decidable.

On the other hand, the Σ_3-fragment of the first order theory of an AC-function symbol, one free function symbol and one constant over the free algebra is shown to be undecidable by R. Treinen [18].

Proposition 4.6 *There exists an instance of combination, namely the combination of the theory of an AC-symbol and the free theory, the Σ_3-fragment of* *which is undecidable, though the full first order theories of both components are decidable.*

5 Free Amalgamation and Negative Constraints: Ground Solvability

To discuss ground solvability, we first have to extend the notion of a ground solution known from equational unification to the more general case of quasi-free structures. In an equational theory, a solution is ground, if no solution term contains a variable. For quasi-free structures, this just means, that the stabiliser of every solution element has to be empty. We also define the notion of a restrictive solution for quasi-free structures. A corresponding notion was introduced by F. Baader & K. U. Schulz in [2] for equational disunification problems. There it said that a solution is restrictive, if whenever a variable is assigned a complex term in the solution, then this term is not equivalent (modulo the equational theory) to a variable.

Definition 5.1 Let Γ be a constraint problem. A solution is called *ground*, iff every solution element has an empty stabiliser.

A solution σ is called *restrictive*, iff for every variable $v \in Var(\Gamma)$ such that $Lab(v) = \Sigma$: $\sigma(v)$ is not an atom.

On the basis of this definition we can now present the main statement in this section.

Theorem 5.2 *Let $(\mathfrak{A}^\Sigma, X)$ and $(\mathfrak{B}^\Delta, X)$ be two quasi-free structures over disjoint signatures with infinitely many ground elements. The existence of a ground solution of mixed $\Sigma \cup \Delta$-constraint problems in the free amalgamated product $\mathfrak{A}^\Sigma \otimes \mathfrak{B}^\Delta$ is decidable, if the existence of restrictive solutions of pure constraint problems with generalised linear constant restrictions is decidable in both components $(\mathfrak{A}^\Sigma, X)$ and $(\mathfrak{B}^\Delta, X)$.*

It would of course be desirable to reduce ground solvability of mixed constraints in the free amalgam to ground solvability in the components. But the decomposition algorithm given in the last section does not permit so. The algorithm is correct, i.e., the existence of ground solutions in the components implies the existence of a ground solution in the free amalgamated product. But unfortunately, the existence of a ground solution in the free amalgamated product does not imply the existence of ground solutions in the components. In [2], F. Baader and K. U. Schulz give an example for equational disunification (4.2 on page 243) that demonstrates this fact. Therefore we have to demand the existence of restrictive solutions in the components in order to get ground solvability in the combined domain.

Proposition 5.3 *Assume that both $\mathfrak{A}^\Sigma$ and $\mathfrak{B}^\Delta$ contain an infinite number of ground elements. The input problem Γ has a ground solution, if and only if there exists a generalised linear constant restriction L such that both $(\Gamma_{3,\Sigma}, L)$ has a restrictive solution in $\mathfrak{A}^\Sigma$ and $(\Gamma_{3,\Delta}, L)$ has a restrictive solution in $\mathfrak{B}^\Delta$.*

6 Independence Properties of Equational Theories

In equational theories, the only negative constraints are of course disequations. For a general introduction to disunification, we refer the reader to [8].

Unitary Theories

Theorem 6.1 *Let E be a unitary equational theory. Then E has the independence of negative constraints property.*

The reason for this theorem is that every solution σ_i of $\Gamma^+ \wedge C_i^-$ is an instance of μ, the most general unifier of Γ^+, and μ is also a solution of $\Gamma^+ \wedge C_i^-$. Hence μ is a unifier of $\Gamma^+ \wedge \Gamma^-$. It is noteworthy that there are non-unitary equational theories which have the independence property. An example is the theory AMh of Abelian Monoids with a homomorphism (see [10]).

Equational Theories without Independence Property

In this section, we present the following statement: If an equational theory is such that one of its unification problems has a minimal complete set of unifiers of finite cardinality, but no single most general unifier, then this theory does not have the independence property. The incomparability of the solutions is used to construct disequations that can be solved individually, but not collectively.

Lemma 6.2 *Let E be a non-unitary equational theory. Let Γ^+ be a unification problem with variables X and $\{\mu_1, \ldots, \mu_n\}$ (where $n > 1$) be a minimal complete set of solutions of Γ^+ with domain X. Then there exists a set of disequations $\{d_1, \ldots, d_n\}$ such that for each $i = 1, \ldots, n$ the problem $\Gamma^+ \wedge d_i$ is solvable, but $\Gamma^+ \wedge \{d_1, \ldots, d_n\}$ is not.*

Theorem 6.3 *Let E be an equational theory that has a unification problem where the cardinality of the minimal complete set of unifiers is finite and larger than 1. Then E has* not *the independence property.*

Corollary 6.4 *Let E be a finitary equational theory. Then E has* not *the independence property.*
The theory A of an associative function symbol and constants has not *the independence property.*

An example of an A-unification problem that has exactly two unifiers is $\{x \circ y \doteq a \circ a \circ a\}$ where $\circ$ is the associative function symbol, x and y variables and a a constant.

Combining Equational Theories and the Independence Property

In this section, we present sufficient conditions under which the independence property of equational theories is preserved under combination. This result seems interesting in its own, since it shows that even for the specific case of equational theories the conditions are quite restrictive, as the following theorem shows.

Theorem 6.5 *Let E and F be two unitary regular and collapse-free theories over disjoint signatures. Then the combined theory $E \cup F$ is again unitary, hence has the independence property.*

Unfortunately, the requirement that E and F be regular and collapse-free[2] cannot be weakened as the following example shows. Consider the theory of Boolean rings, which is not regular. It is known (see [4]) that in this theory, unification with constants is unitary. But general unification is finitary. And in any theory, general unification can be regarded as an instance of combining unification with constants in that theory with syntactic unification.

The above demanded requirements are rather restrictive. There are only two "natural" theories which come to mind that are both unitary and regular and collapse-free. One is of course syntactic unification. And the other is single-sided distributivity, such as distributivity to the left ($D_L = \{f(g(x, y), z) = g(f(x, z), f(y, z))\}$) and distributivity to the right.

7 Independence Properties of Quasi-free Structures

Before we can start to extend the results of the previous sections to quasi-free structures, it is necessary to generalise some basic notions of unification theory to quasi-free structures. Remember that our constraint problems are existentially quantified conjunctions of literals.

Definition 7.1 Let Γ be a positive constraint problem of the quasi-free structure $(\mathfrak{A}^{\Sigma}, X)$. Let σ and τ be two solutions of Γ. We say σ is *more general* than τ, or τ is an instance of σ, with respect to the variables in Γ (and write $\sigma \leq_{\Gamma} \tau$), iff there exists an endomorphism $m \in End_{\mathfrak{A}}^{\Sigma}$ such that for all $x \in Var(\Gamma)$ we have $m \circ \sigma(x) = \tau(x)$.

A quasi-free structure $(\mathfrak{A}^{\Sigma}, X)$ is called *unitary*, iff for every solvable positive constraint problem Γ there exists a solution μ such that every solution of Γ is an instance of μ.

Examples of unitary quasi-free structures are unitary equational theories, since all instantiating substitutions are indeed endomorphisms. Rational tree algebras are also unitary.

[2] An equational theory is *regular*, if in each axiom the left and the right side contain the same set of variables, and collapse-free, if it does not contain an axiom $x = t$ where x is a variable and t a non-variable term.

Theorem 7.2 *Let* $(\mathfrak{A}^\Sigma, X)$ *be a unitary quasi-free structure. Then* $(\mathfrak{A}^\Sigma, X)$ *has the independence property.*

It is worth mentioning that G. Smolka and R. Treinen [17] give an example of a quasi-free structure which is not an equational theory, is not unitary, but has the independence property, namely the algebra of feature trees with arity. This structure is interesting, because it has unique syntactic solved forms, although it is not unitary. This is a consequence of the sort system chosen by the authors. With choice of a different sort system (a flat hierarchy) the structure would be unitary.

To lift the modularity result, we also have to generalise the notions "collapse-free" and "regular" to quasi-free structures.

Definition 7.3 A quasi-free structure $(\mathfrak{A}^\Sigma, X)$ is called *non-collapsing*, iff every endomorphism maps non-atoms to non-atoms, i.e., $m(a) \in A \setminus X$ for all $m \in End_{\mathfrak{A}}^{\Sigma}$ and all $a \in A \setminus X$.

In order to define the notion *regular* for quasi-free structures, we observe the following.

Lemma 7.4 *Let* $(\mathfrak{A}^\Sigma, X)$ *be a quasi-free structure,* $m \in End_{\mathfrak{A}}^{\Sigma}$ *an endomorphism, and* $a \in A$ *some element. Suppose* $\mathrm{Stab}^{\mathfrak{A}}(a) = \{x_1, \ldots x_k\}$. *Then* $\mathrm{Stab}^{\mathfrak{A}}(m(a)) \subseteq \bigcup_{i=1}^{k} \mathrm{Stab}^{\mathfrak{A}}(m(x_i)) = \mathrm{Stab}^{\mathfrak{A}}(m(\mathrm{Stab}^{\mathfrak{A}}(a)))$.

Definition 7.5 A quasi-free structure $(\mathfrak{A}^\Sigma, X)$ is *regular*, iff for all $m \in End_{\mathfrak{A}}^{\Sigma}$ and all $a \in A : \mathrm{Stab}^{\mathfrak{A}}(m(a)) = \mathrm{Stab}^{\mathfrak{A}}(m(\mathrm{Stab}^{\mathfrak{A}}(a)))$.

Now, we can give sufficient conditions for the component structures in order for the free amalgamated product to have the independence property.

Theorem 7.6 *Let* $(\mathfrak{A}^\Sigma, X)$ *and* $(\mathfrak{B}^\Delta, X)$ *be two unitary regular non-collapsing quasi-free structures over disjoint signatures. Then the free amalgamated product* $\mathfrak{A}^\Sigma \otimes \mathfrak{B}^\Delta$ *is again unitary, hence has the independence property.*

The theorem is based on the fact that there exists a *deterministic* combination algorithm for unitary regular and non-collapsing quasi-free structures, which was developed for the more specific case of the combination of equational theories by K. U. Schulz in [15]. This algorithm computes most general solutions (see [10]).

8 Conclusion

We analysed the role of negation in the combination of quasi-free structures. The contribution of this paper is twofold. In the first half, we showed that the existential fragment of the free amalgamated product $\mathfrak{A}^{\Sigma} \otimes \mathfrak{B}^{\Delta}$ is decidable provided that the solvability of conjunctions of literals with linear constant restrictions is decidable in both components $\mathfrak{A}^{\Sigma}$ and $\mathfrak{B}^{\Delta}$. The existence of a ground solution for a mixed $\Sigma \cup \Delta$ constraint problem in the free amalgamated product is decidable, if the existence of restrictive solutions of conjunctions of literals with linear constant restrictions is decidable in the components. This shows that the free amalgamated product is a suitable combined solution domain for combining constraint systems which have negation in their languages. The reader familiar with Rational Amalgamation, another general combination method introduced in [11], may be interested to hear that Rational Amalgamation seems not very well suitable for combining constraint systems with negation (see [10]).

We also investigate the independence of negative constraints property in quasi-free structures and the free amalgamated product. Looking at the particular subclass of equational theories, we found that unitary theories have the independence property while finitary do not. We saw, too, that the union of two signature-disjoint, unitary, regular and collapse-free equational theories is again unitary and hence has the independence property. For the general case of quasi-free structures, we showed that unitary quasi-free structures have the independence property. We also obtained a modularity result which states that the free amalgamated product of two signature-disjoint, unitary, regular and non-collapsing quasi-free structures is again unitary, and thus has the independence property.

Acknowledgement

The author would like to thank in particular Klaus U. Schulz for suggestions and comments on drafts of this paper and Franz Baader for discussing some side aspects.

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