On Monadic Second-Order Theories of Multidominance Structures

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1 Introduction

Multidominance structures were introduced by Kracht [4] to provide a data structure for the formalisation of various aspects of GB-Theory. Kracht studied the PDL-theory of MDSes and showed that this theory is decidable in [5], actually 2EXPTIME-complete. He continues to conjecture that thus the MSO-theory of MDSes should be decidable, too. We show here the contrary. Actually, both the MSO-theory over vertices only and the MSO-theory over vertices and edges turn out to be undecidable.

2 Preliminaries

Graphs

All graphs and structures discussed in this paper are *finite*.

An undirected graph is a pair $G = (V_G, E_G)$ where V_G is a finite set of vertices and E_G is a set of edges, a subset of $V_G \times V_G$. In the graphs (and multi graphs) we consider there is *never* an edge from any vertex to itself.

A graph signature Σ is a finite set of symbols together with an arity. In this paper we do not consider hypergraphs. Consequently, all symbols have arity 0 for vertex labels or 2 for edge labels. Thus we split Σ into Σ_0 and Σ_2. Edge labels are sometimes also called colours. For the purpose of this paper we are mainly interested in the *structure* of graphs. We thus assume that Σ_0 consists of a single blank symbol, which is suppressed in the following.

A multi graph of signature Σ is a quaduple $G = (V_G, E_G, sig, inc)$ where V_G is a finite set of vertices, E_G is a finite set of directed edges, $sig : V_G \to \Sigma_0 \cup E_G \to \Sigma_2$ a function assigning each vertex a label and each edge a colour, and $inc : E_G \to V_G \times V_G$ a function assigning each edge its starting and ending vertices.

Note that there may be more than one edge between two vertices of a multi graph. A multi graph is *simple*, iff each pair of vertices is connected by at most one edge. Paths in a multi graph are uncoloured but directed. A multi graph is *acyclic* iff no path connets a vertex with itself. A multi graph is *rooted* iff there is a vertex r such that (i) there is no vertex v and edge e with $inc(e, v, r)$ and (ii) every vertex is reachable from r.

The underlying undirected graph G_u of a multi graph $G = (V_G, E_G, sig, inc)$ is $G_u = (V_G, E_u)$ where $(v, w) \in E_u$ iff there is some $e \in E_G$ with $inc(e, v, w)$ or $inc(e, w, v)$. That is we forget about the direction of edges and multiple edges between two vertices are reduced to one.

The *complete* graph K_k is an undirected graph $G = (V, E)$ where $V = \{1, \ldots, k\}$ and for all $1 \leq i, j, \leq k$ holds $(i, j) \in E$ iff $i \neq j$, i.e., there is an edge between each pair of different vertices. Figure 1 shows the complete graphs K_3, K_4, K_5.

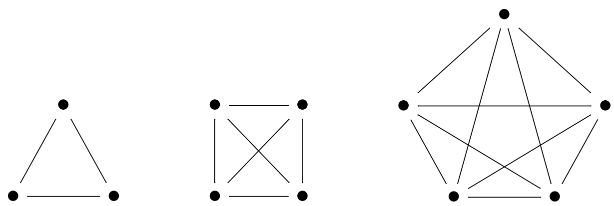

Fig. 1. The complete graphs K_3, K_4, and K_5.

Tree width is a notion introduced by Robertson and Seymour [6] to measure how similar to a tree a graph is. It assigns each graph a natural number, where smaller means closer to a tree. The number can be interpreted as the maximal number of independent paths between any two vertices. The following formal definition works independently of whether a graph is directed or not.

Definition 1. *A* tree decomposition *of a multi graph $G = (V, E, sig, inc)$ is a pair (T,S), where T is an unordered tree and S is a family of sets indexed by the vertices of T such that*

1. $\bigcup_{X_t \in S} X_t = V$.
2. *For all $e \in E$ there is a unique $X_t \in S$ such that if $inc(e, v, w)$ then $v, w \in X_t$.*
3. *For all $v \in V$ the subgraph of T induced by $\{t \mid v \in X_t\}$ is connected.*

The width *of such a decomposition is $\max_{X_t \in S} |\{v \mid v \in X_t\}| - 1$, i.e., the largest number of vertices in a single set of the decomposition minus 1.*
A graph G is of tree width *k if and only if the smallest width of a tree decomposition of G is k.*

We will make use of the following two simple observations known from graph theory.

Lemma 1. 1. *The complete graph K_k has tree width $k - 1$.*
2. *Let $G = (V, E, sig, inc)$ be a multi graph of tree width k. Then its underlying undirected graph G_u has the same tree width k.*

The following notion of a graph minor was also introduced and extensively studied by Robertson and Seymour.

Definition 2. *A graph G is a minor of $H = (V, E, sig, inc)$ if it is the result of applying a finite sequence of the following three operations.*

- *Edge deletion*
  If e is an edge, then this operation removes e from the graph.
- *Vertex deletion*
  If v is an unconnected vertex, then this operation removes v from the graph.
- *Edge contraction*
  Let e be an edge and $inc(e, v, w)$. Then edge e and vertex w are removed from the graph and each occurrence of w in inc is replaced by v. In effect, vertices v and w are fused.

The tree width of a graph minor provides a lower bound for the tree width of a graph.

Lemma 2. ([1], Lemma 16) *If G is a minor of H, then tree width(G) $\leq$ tree width(H).*

Monadic second-order logic of graphs

When talking about monadic second-order theories of graphs one distinguishes whether quantification is restricted to vertices or quantification is applicable to vertices and edges. The first theory is usually denoted $\mathcal{MS}_1$, the second $\mathcal{MS}_2$. It is known that definability of and decidability over certain classes of finite graphs vary depending on whether $\mathcal{MS}_1$ or $\mathcal{MS}_2$ is considered.

The $\mathcal{MS}_1$-theory has vertex variables only, there are individual and set variables. For each vertex label $L \in \Sigma_0$ and individual variable x there is an atomic formala $L(x)$. For each edge colour $C \in \Sigma_2$ and pair of individual variables x, y there is an atomic formula $C(x, y)$. Furthermore equality and set membership are atomic. Complex formulae are constructed by boolean connectives and first-order and set existential and universal quantification.

More precisely let $\mathcal{X}_0 = \{x_0, x_1, x_2, \dots\}$ be a denumerably infinite set of first-order vertex variables and $\mathcal{X}_1 = \{X_0, X_1, X_2, \dots\}$ be a denumerably infinite set of vertex set variables. Atomic formulae are

- $L(x)$ for each vertex label $L \in \Sigma_0$,
- $C(x, y)$ for each edge colour $C \in \Sigma_2$,
- $x = y$,
- $x \in X$,
- $X = Y$.

Complex formulae are created by boolean operations and quantification: (let ϕ and ψ be formulae)

- $\neg\phi$, $\quad \phi \wedge \psi$, $\quad \phi \vee \psi$,
- $\forall x\phi$, $\quad \exists x\phi$,
- $\forall X\phi$, $\quad \exists X\phi$.

The sematics is the usual one.

The $\mathcal{MS}_2$-theory has two sorts of variables, namely vertex variables and edge variables. For both sorts there are individual and set variables. For each vertex label $L \in \Sigma_0$ and individual vertex variable x there is an atomic formala $L(x)$. For each edge colour $C \in \Sigma_2$, individual edge variable e and pair of individual vertex variables x, y there is an atomic formula $inc_C(e, x, y)$. Furthermore equality and set membership for both sorts of variables are atomic. Complex formulae are constructed by boolean connectives and first-order and set existential and universal quantification for both sorts of variables.

Let $\mathbf{V}$ be the sort of vertices and $\mathbf{E}$ be the sort of edges. Let $\mathcal{X}_0 = \{x_0, x_1, x_2, \dots\}$ be a denumerably infinite set of first-order variables of sort $\mathbf{V}$ and $\mathcal{X}_1 = \{X_0, X_1, X_2, \dots\}$ be a denumerably infinite set of set variables of sort $\mathbf{V}$. Let $\mathcal{E}_0 = \{e_0, e_1, e_2, \dots\}$ be a denumerably infinite set of first-order variables of sort $\mathbf{E}$ and $\mathcal{E}_1 = \{E_0, E_1, E_2, \dots\}$ be a denumerably infinite set of set variables of sort $\mathbf{E}$. The atomic formulae are

- $L(x)$ for each vertex label $L \in \Sigma_0$,
- $inc_C(e, x, y)$ for each edge colour $C \in \Sigma_2$,
- $x = y$,
- $x \in X$,
- $X = Y$,
- $e_0 = e_1$,
- $e \in E$,
- $E_0 = E_1$.

Quantification can now be applied to vertices and edges. More precisely the complex formulae are constructed as follows (where ϕ and ψ are formulae):

- $\neg\phi$, $\quad \phi \wedge \psi$, $\quad \phi \vee \psi$,
- $\forall x\phi$, $\quad \exists x\phi$,
- $\forall X\phi$, $\quad \exists X\phi$,
- $\forall e\phi$, $\quad \exists e\phi$,
- $\forall E\phi$, $\quad \exists E\phi$.

The sematics is the usual one.

3 Multidominance Structures

We introduce multidominance structures in this section quoting the relevant definitions from [5]. MDSes are structures which can be seen as being derived from binary trees[1]. As binary trees, they are rooted directed graphs where each vertex has either 0 or 2 immediate successors. The graph may not contain a loop. In difference to trees, a vertex may have more than one immediate predecessor. The set of immediate predecessors is linearly ordered.

[1] We will later on see that they differ substantially from trees in important ways.

Technically – and we follow here the description by Kracht [5] – the symbol $\succ$ defines an immediate dominance relation, where $x \succ y$ is read as x dominates y. Its inverse is denoted by $\prec$. Nodes are downward binary branching, they have at most two children. The following text is a longer quote from [5].

To implement this we shall assume two relations, $\succ_0$ and $\succ_1$ each of which is a partial function, and $\succ = \succ_0 \cup \succ_1$. We do not require the two relations to be disjoint.

Recall the definition of the transitive closure R^+ of a binary relation $R \subseteq U \times U$ over a set U. It is the least set S containing R such that if $(x, y) \in S$ and $(y, z) \in S$ then also $(x, z) \in S$. Recall that R is loop free if and only if R^+ is irreflexive. Also, $R^* := \{(x, x) \mid x \in U\} \cup R^+$ is the reflexive, transitive closure of R.

Definition 3. *A* preMDS *is a structure $\langle M, \succ_0, \succ_1\rangle$, where the following holds (with $\succ = \succ_0 \cup \succ_1$):*

(P1) If $y \succ_0 x$ and $y \succ_0 x'$ then $x = x'$.

(P2) If $y \succ_1 x$ and $y \succ_1 x'$ then $x = x'$.

(P3) If $y \succ_1 x$ then there is a z such that $y \succ_0 z$.

(P4) There is exactly one x such that for no y, $y \succ x$ (this element is called the root*).*

(P5) $\prec^+$ is irreflexive.

(P6) The set $M(x) := \{y : x \prec y\}$ is linearly ordered by $\prec^+$.

We call a pair $\langle x, y\rangle$ such that $x \prec y$ a *link*. We shall also write $x; y$ to say that $\langle x, y\rangle$ is a link. An MDS is shown in Figure 2. The lines denote the immediate daughter links. For example, there is a link from a upward to c. Hence we have $a \prec c$, or, equivalently, $c \succ a$. We also have $b \prec a$. We use the standard practice of making the order of the daughters implicit: the leftward link is to the daughter number 0. This means that $a \prec_0 c$ and $b \prec_1 c$. Similarly, it is seen that $b \prec_1 d$ and $b \prec_1 h$, while $c \prec_0 d$ and $g \prec_0 h$. It follows that $M(a) = \{c\}$, while $M(b) = \{c, d, h\}$. A link $\langle x, y\rangle$ such that y is minimal in $M(x)$ is called a *root link*. For example, $\langle b, c\rangle$ is a root link, since $c \prec^+ d$ and $c \prec^+ h$. A link that is not a root link is called *derived*. A *leaf* is a node without daughters.

For technical reasons we shall split $\prec_0$ and $\prec_1$ into two relations each. Put $x \prec_{00} y$ iff (= if and only if) $x \prec_0 y$ and y is minimal in $M(x)$; and put $x \prec_{01} y$ iff $x \prec_0 y$ but y in not minimal in $M(x)$. Alternatively, $x \prec_{00} y$ if $x \prec_0 y$ and $\langle x, y\rangle$ is a root link. Let $x \prec_{01} y$ iff $x \prec_0 y$ but not $x \prec_{00} y$. Then by definition $\prec_{00} \cap \prec_{01} = \varnothing$ and

$$\prec_0 = \prec_{00} \cup \prec_{01}$$

Similarly, we decompose $\prec_1$ into

$$\prec_1 = \prec_{10} \cup \prec_{11}$$

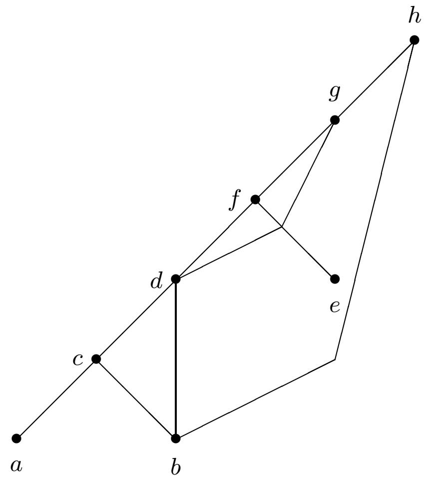

Fig. 2. An MDS (from [5]).

where $x \prec_{10} y$ iff $x \prec_1 y$ and y is minimal in $M(x)$ (or, equivalently, $\langle x, y \rangle$ is a root link). And $x \prec_{11} y$ iff $x \prec_1$ and y is not minimal in $M(x)$. We shall define

$$\prec_{\bullet 0} := \prec_{00} \cup \prec_{10}$$
$$\prec_{\bullet 1} := \prec_{01} \cup \prec_{11}$$

We shall spell out the conditions on these four relations in place of just $\prec_0$ and $\prec_1$. The structures we get are called *MDSs.*

Definition 4. *An* MDS *is a structure* $\langle M, \succ_{00}, \succ_{01}, \succ_{10}, \succ_{11} \rangle$ *which, in addition to (P1) – (P6) of Definition 3 satisfies*

(P7) If $y \in M(x)$ *then* $x \prec_{\bullet 0} y$ *iff* $x; y$ *is a root link (iff* y *is the least element of* $M(x)$ *with respect to* $\prec^+$*).*

This ends the quote from [5]. We observe the following peculiarity. With MDSes defined the way above it is possible that there are two nodes x and y such that $x \succ_0 y$ and $x \succ_1 y$, i.e., y is the left *and* right child of x. This is a deliberate decision on Kracht's side, and justified by the logic to describe MDSes chosen by Kracht. But this decision causes technical problems to us that we will mention later. Since there is hardly any linguistic justification for making a node a left and right child simultanueously, we will also consider simple MDSes. These are MDSes that are simple in the graph theoretic sense of the word.

4 $\mathcal{MS}_1$-Axiomatisation of MDSes

The axiomatisability of MDSes (and regular ones) in $\mathcal{MS}_1$ follows immediately from the axiomatisation of MDSes in PDL that Kracht provides in [5] and the well known fact that PDL formulae can be translated into equivalent $\mathcal{MS}_1$ formulae. But $\mathcal{MS}_1$ bing a powerful logic we can easily translate the defining properties of MDSes directly into $\mathcal{MS}_1$.

(1) $y \succ_0 x \wedge y \succ_0 x' \Longrightarrow x = x'$.
(2) $y \succ_1 x \wedge y \succ_1 x' \Longrightarrow x = x'$.
(3) $y \succ_1 x \Longrightarrow \exists z\, y \succ_0 z$.
(4) $\exists x\, (\nexists y\, y \succ x) \wedge (\forall z\, (\nexists y\, y \succ z) \Longrightarrow z = x)$.
(5) $\nexists x\, x \succ^+ x$.
(6) $\exists E\, (y \succ x \Longleftrightarrow y \in E) \wedge \forall z, z'\, (z \in E \wedge z' \in E) \Longrightarrow (z \succ^+ z' \vee z' \succ^+ z)$.
(7) $\exists E\, (y \succ x \Longleftrightarrow y \in E) \wedge$
$\exists y\, y \in E \wedge y \succ_{\bullet 0} x \wedge \forall z\, z \in E \Longrightarrow (z = y \vee z \succ^+ y) \wedge \forall z\, z \in E \wedge z \neq y \Longrightarrow$
$z \succ_{\bullet 1} x$,
(8) $x \succ_0 y \wedge x \succ_1 z \Longrightarrow y \neq z$.

Remember that $\mathcal{MS}_1$ is capable of defining the transitive closure of an $\mathcal{MS}_1$-definable binary relation. We skip this definition here (refering the reader to, e.g., [2]) and just use $\succ^+$ as its abbreviation.

Finiteness of the MDSes is also $\mathcal{MS}_1$-defineable. We will not present this fact in details. Rather we explain the method to be used. One can define a linear order on all the vertices extending $\succ$. This order has to be discrete and has to have a maximal element. One defines a successor relation on the order and demands that the maximal element is in the set of elements reachable from the minimal element, which is the root, via the transitive closure of the successor relation.

5 Undecidability of the $\mathcal{MS}_2$-Theory of MDSes

For showing the undecidability of the $\mathcal{MS}_2$-theory of MDSes we use the following strategy. We define a sequence $(G_k)_{k\in\mathbb{N}}$ of MDSes such that each element in the sequence contains its predecessor as a subgraph and each element G_i has the complete graph K_i as a graph minor. This way we can show that the class of MDSes has unbounded tree width. We then use a criterion by Seese [7] to deduce that its $\mathcal{MS}_2$ theory is undecidable.

Definition 5. *Recursively define a sequence $(G_k)_{k>1}$ of MDSes as follows. Define $G_2 = (\{1.1, 2.1\}, \{(2.1, 1.1)\}, \emptyset, \emptyset, \emptyset)$. For $k > 2$ set*

- $V_k = V_{k-1} \cup \{k.i \mid i = 1 \ldots k-2\}$,
- $\succ_{00_k} = \succ_{00_{k-1}} \cup \{(k.i, k.i+1) \mid i = 1 \ldots k-3\} \cup \{(k.k-2, k-1.1)$,
- $\succ_{01_k} = \succ_{01_{k-1}} = \emptyset$,
- $\succ_{10_k} = \succ_{10_{k-1}} = \emptyset$,

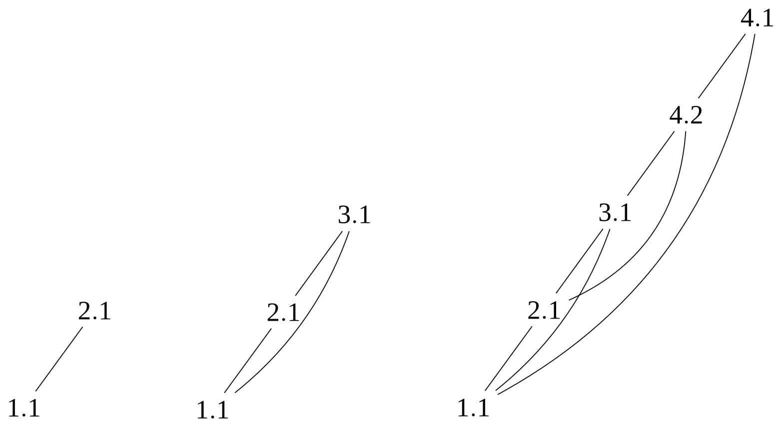

Fig. 3. The MDSes G_2, G_3 and G_4

– $\succ_{11_k} = \succ_{11_{k-1}} \cup \{(k.i, i.1) \mid i = 1 \ldots k-2\}$,
– $G_k = (V_k, \succ_{00_k}, \succ_{01_k}, \succ_{10_k}, \succ_{11_k})$.

The MDSes G_2, G_3 and G_4 are depicted in Figure 3. The MDSes G_5 and G_6 – being rather large – are depicted in Appendix A. It is immediately obvious that G_{k-1} is a subgraph of G_k. We quickly check that G_k is indeed an MDS.

Lemma 3. *For $k > 1$ each graph G_k is a simple MDS.*

Proof. Conditions (P1), (P2), and (P4) (rootedness) are obviously true for all G_k. None of the graphs contains a loop (P5). And simplicity (P8) is also observed. The thing to be shown are conditions (P6) and (P7) stating that the set of parents of each node is linearly ordered and that the lowest element in the order is the only root link.

Observe that for each G_k all nodes are linearly ordered by $\succ_{00_k}$ and that by definition of $\succ_{00_k}$ each node different from the root has a single root link. A node $(i.j)$ with $j > 1$ has a single parent, which is $(i.j-1)$. For $i > 1$ all nodes $(i.1)$ have set $M(i.1) = \{(l.1) \mid i+1 < l \leq k\} \cup \{(i+1, i-1)\}$. For node (1.1), $M(1.1) = \{(l.1) \mid i+1 \leq l \leq k\}$. $\square$

Lemma 4. *For $k > 1$ each MDS G_k contains the complete graph K_k as a minor.*

Proof. For $k = 2, 3$ the complete graph K_k is the undirected version of G_k.
For $k > 3$ the lemma is shown by induction on k. The general method is to contract all edges that connect vertices with the same main address.
For $k = 4$ contract $4.1 \succ_{00_4} 4.2$. As there is an edge from 4.1 to 1.1 and one each from 4.2 to 3.1 and 2.1 the undirected graph after contraction is K_4.

For $k > 4$ contract the set of edges $\{k.i \succ_{00_k} k.i+1 \mid 1 \leq i \leq k-3\}$. As a result the set of vertices $\{k.i \mid 1 \leq i \leq k-2\}$ is fused to a single vertex k.
By induction hypothesis, the subgraph G_{k-1} can be contracted to K_{k-1}. Now, since there is an edge from $k.i$ to $i.1$ for each $1 \leq i \leq k-2$ and an edge from

$k.k-2$ to $k-1.1$ (by definition of G_k), each vertex in K_{k-1} is connected to some vertex in $\{k.i \mid 1 \le i \le k-2\}$. After fusing these into a single vertex k the resulting graph is thus K_k. □

Theorem 1. *The $\mathcal{MS}_2$-theory of the classes of MDSes and simple MDSes is undecidable.*

Proof. As a consequence of the above lemma and Lemma 2 the classes of MDSes and simple MDSes have unbounded tree width.
Seese ([7], Theorem 8) showed that if a class of graphs has a decidable $\mathcal{MS}_2$-theory, it has bounded tree width. □

6 Equivalence of $\mathcal{MS}_1$ and $\mathcal{MS}_2$ on MDSes

The aim of this section is to show that $\mathcal{MS}_1$ has the same expressive power over MDSes as $\mathcal{MS}_2$. In other words, the option of edge set quantification does not extend the expressive power of MSO on MDSes. To show this we use a criterion by Courcelle. He showed in [3] that for uniformly k-sparse classes of simple graphs the two logics $\mathcal{MS}_1$ and $\mathcal{MS}_2$ have the same expresive power. A class of graphs is uniformly k-sparse if for some fixed k the number of edges of each subgraph of a graph is at most k-times the number of vertices.

Definition 6. *A finite multi graph G is* k-sparse*, if there is some natural number k such that $Card(E_G) \le k\,Card(V_G)$. A finite multi graph G is* uniformly k-sparse *if each subgraph of G is k-sparse. A class of finite multi graphs is* uniformly k-sparse *if there is some natural number k such that each multi graph of the class is uniformly k-sparse.*

On the base of the following little lemma it is easy to see that MDSes are uniformly 2-sparse.

Lemma 5. *Let G be a multi graph.*
If the maximal in degree of G is d then G is uniformly d-sparse.
If the maximal out degree of G is d then G is uniformly d-sparse.

Proof. We can count edges by counting end points or starting points of edges. I.e.

$$Card(E_G) = \sum_{v \in V_G} indeg(v) = \sum_{v \in V_G} outdeg(v).$$

If the maximum in (out) degree is d the above equation can be approximated by

$$Card(E_G) \le d\,Card(V_G).$$

See also [3], Lemma 3.1. □

Corollary 1. *The class of MDSes is uniformly 2-sparse.*

Proof. MDSes share with binary trees the property of having a maximal out degree of 2. □

Thus simple MDSes fulfil the criterion set out in [3].

Proposition 1. *The logics $\mathcal{MS}_1$ and $\mathcal{MS}_2$ have the same expressive power over the class of simple MDSes.*

Proof. By Theorem 5.1 of [3], the same properties of multi graphs are expressible by $\mathcal{MS}_1$ and $\mathcal{MS}_2$ formulae for the class of finite simple 2-sparse multi graphs. □

Corollary 2. *The $\mathcal{MS}_1$-theory of the class of simple MDSes is undecidable.*

Proof. Follows immediately from the above proposition and Theorem 1. □

The restriction to simple MDSes can be overcome on the basis of the following observation. Since we have only four colours of edges, simplicity can be defined in first-order logic. The following axiom does this.

$$\forall x, y\, (x \succ_{00} y \vee x \succ_{01} y) \Longleftrightarrow \neg(x \succ_{10} y \vee x \succ_{11} y).$$

Theorem 2. *The $\mathcal{MS}_1$-theory of the class of MDSes is undecidable.*

Proof. Suppose the $\mathcal{MS}_1$-theory of the class of MDSes were decidable. Add the above axiom of simplicity to gain a decision procedures for the $\mathcal{MS}_1$-theory over simple MDSes. This contradicts Corollary 2. □

Theorem 3. *The logics $\mathcal{MS}_1$ and $\mathcal{MS}_2$ have the same expressive power over the class of MDSes.*

Proof. Both theories have the same degree of undecidability. □

7 Conclusion

We showed that both the $\mathcal{MS}_1$-theory and the $\mathcal{MS}_2$-theory over MDSes are undecidable – contrary to what Kracht conjectured. There was a good reason for Kracht's conjecture, namely $\mathcal{MS}_1$ is not much more powerful than PDL. So, how can this result be interpreted. We'd like to propose the following view. Courcelle showed that the property of being a minor is definable by an MSO-definable transduction. But this property is *not* PDL-definable. It is not possible to code grids in a direct way with MDSes, basically because any set of parents is linearly ordered by dominance. But grids can be minors of MDSes.

There is the question whether we can find a natural restriction on MDSes to bound their tree width to regain decidability of the MSO-theories. It is of course possible to just demand this or enforce it by e.g., demanding MDSes to be generable by context-free graph grammars. But these restrictions do not seem to have a motivation different from bounding the tree width and thus seem arbitrary. It would be much nicer if restrictions could be found that relate multi dominance to restrictions for movement.

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A MDSes G_5 and G_6

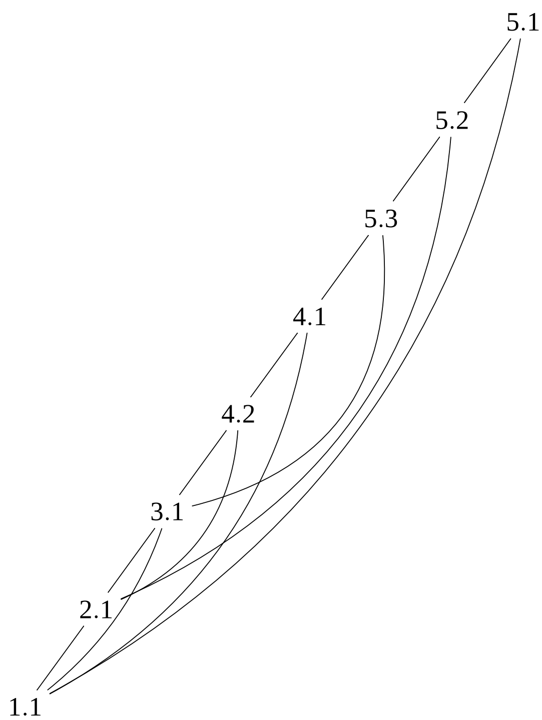

Fig. 4. MDS G_5

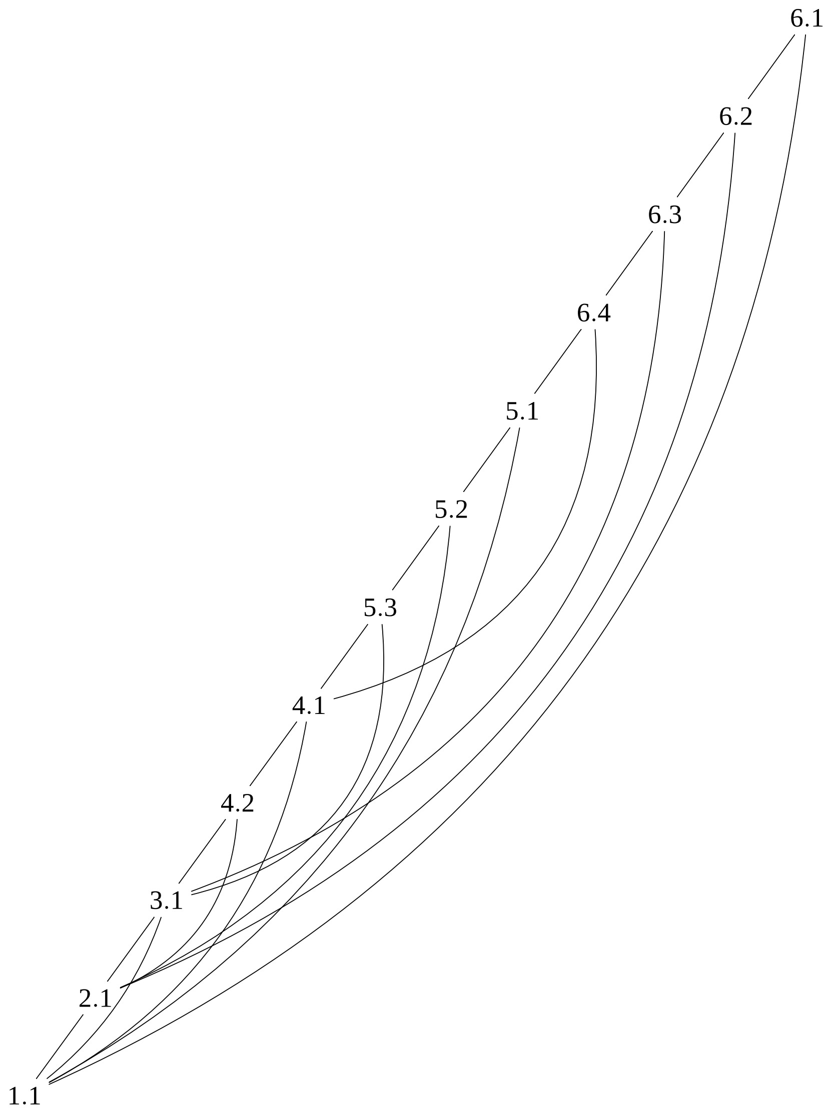

Fig. 5. MDS G_6