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Adjunction as Substitution
An Algebraic Formulation of Regular, Context-Free and Tree Adjoining Languages

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1 Introduction

There have been many attempts to give a coherent formulation of a hierarchical progression that would lead to a refined partition of the vast area stretching from the context-free to the context-sensitive languages. The purpose of this note is to describe a theory that seems to afford a promising method of interpreting the tree adjoining languages as the natural third step in a hierarchy that starts with the regular and the context-free languages.

The formulation of this theory is inspired by two sources: the categorical concept of an algebraic theory and the powerful tool of macro variables which is well established within the framework of program schemes. The rough idea is that according to their intended interpretation objects of algebraic theories are sets of derived operations and that macro variables range over these sets. Guided by this conception we show how monadic macro variables provide a perspective from which both the context-free and the tree adjoining languages become realizations of the same general notion when the relevant underlying algebra is specified. The context-free languages are determined by an inductive process in which monadic macro variables are replaced by derived operations of an algebra all of whose operations are unary. In the case of the tree adjoining languages the same substitution process is applied to derived operations over an arbitrary algebra.

The central notion in this account is that of a higher-order substitution. Whereas in traditional presentations of rule systems for language families the emphasis has been on a first-order substitution process in which auxiliary variables are replaced by elements of the carrier of the proper algebra - concatenations of terminal and auxiliary category symbols in the string case - we lift this process to the level of operations defined on the elements of the carrier of the algebra. Our own view is that this change of emphasis provides the adequate platform for a better understanding of the operation of adjunction. To put it in a nutshell: Adjoining is not a first-order, but a second-order substitution operation.

This is not the first place that macro productions are put to use outside the field of program schemes. It has been known since the pioneering work of M.J. Fischer that macro grammars (without the restriction to monadic variables) are weakly equivalent to indexed grammars. The paper by Maibaum also contains one direction of the equivalence between context-free grammars and monadic extensions of regular grammars. The other direction seems to be part of the folklore wisdom pertaining to tree language theory.

The greatest advantage that comes with a macro-like presentation derives from an explicit notation for derived operators in an algebra. As was emphasized above, macro variables range over complex or derived operations. These derived operations are formed by functional composition, starting from the primitive operations. There is an alternative conception of this compositionally closed family of functions as an algebra whose carrier is the disjoint union of the derived operations and whose only primitive operations are just the suitably typed composition functions. This conception views operations as elements of a many-sorted algebra. The corresponding shift on the symbolic level "nominalizes" the second-order macro variables and maps them into object variables. As a result of this translational shift a system with macro-like productions becomes an instance of a regular tree grammar. This latter grammar format provides the class of structures that fall under the purview of logical techniques that have been successfully applied to problems surrounding the metatheory of several linguistic models. The algebraic approach sketched above will enable us to apply these techniques to structural properties of grammatical formalisms that properly include the range of mildly context-sensitive devices.

In the following we will first introduce the basic notion of a context-free tree grammar (Section 2). Then, in Section 3, the equivalence between context-free string languages and context-free tree languages over monadic alphabets is established. Section 4 is devoted to the presentation of the tree adjoining languages as languages generated by context-free tree grammars where all the functional symbols are monadic. The concluding section points to the model theoretic techniques that are applicable to the kinds of tree producing systems underlying the constructions in this paper.

2 Preliminaries

The purpose of this section is to fix notations and to present definitions related to tree grammars.

Definition 1. A **ranked alphabet** or **signature** Σ is an indexed family $\Sigma_n (n \in \mathbb{N})$ of disjoint sets. A symbol in Σ_n is called an operator of **rank** n. If $n = 0$ then $f \in \Sigma_0$ is also called a **constant**.

Example 1.

a) $\Sigma_0 = \{\varepsilon\} \cup V \qquad \Sigma_2 = \{\frown\}$
Single–sorted signature of semi–groups, extended by a finite set of constants V.

b) $\Sigma_0 = \{\varepsilon\} \qquad \Sigma_1 = \{a \mid a \in V\}$
Single–sorted signature of a *monadic* algebra.

Definition 2. For a ranked alphabet Σ, we denote by $T(\Sigma)$ the set of **trees** over Σ. $T(\Sigma)$ is inductively defined as follows:

(i) $\Sigma_0 \subseteq T(\Sigma)$

(ii) If $f \in \Sigma_n$ and $t_i \in T(\Sigma)$ for $i = 1, \dots, n$
$f(t_1, \dots, t_n) \in T(\Sigma)$

We fix an indexed set $X = \{x_1, x_2, \dots\}$ of *variables* and denote by X_n the subset $\{x_1, \dots, x_n\}$. Variables are considered to be constants, i.e. operators of rank 0. For a ranked alphabet Σ the family $T(\Sigma, X)$ is defined to be $T(\Sigma(X))$, where $\Sigma(X)$ is the ranked alphabet with $\Sigma(X)_0 = \Sigma_0 \cup X$ and $\Sigma(X)_n = \Sigma_n$ for every $n \neq 0$. For $Y \subseteq X$ a tree $t \in T(\Sigma, X)$ is called *linear* in Y if no variable in Y has more than one occurrence in t. A subset L of $T(\Sigma)$ is called a *tree language* over Σ.

Having described the tree terms, it remains to specify the central notion of an algebra and to give a precise definition of the way in which the operator symbols induce an operation on an algebra.

Definition 3. Suppose that Σ is a ranked alphabet. A Σ-**algebra** $\mathbb{A}$ is a pair $\mathbb{A} = (A, (f_{\mathbb{A}})_{f \in \Sigma})$ where the set A is the **domain** or **carrier** of the algebra and for each operator $f \in \Sigma_n$, $f_{\mathbb{A}} : A^n \to A$ is an operation of arity n on A.

Different algebras, defined over the same operator domain, are related to each other if there exists a mapping between their carriers that is compatible with the basic structural operations.

Definition 4. A Σ-**homomorphism** of Σ-algebras $h : \mathbb{A} \longrightarrow \mathbb{B}$ is a function $h : A \longrightarrow B$, such that for every operator f of rank n

$$h(f_{\mathbb{A}}(a_1, \dots, a_n)) = f_{\mathbb{B}}(h(a_1), \dots, h(a_n))$$

for every n-tuple $(a_1, \dots, a_n) \in A^n$.

The set of trees $T(\Sigma, X)$ can be made into a Σ-algebra by defining the operations in the following way. For every f in Σ_n, for every $(t_1, \dots, t_n)$ in $T(\Sigma, X)^n$

$$f_{\mathbb{T}(\Sigma,X)}(t_1, \dots, t_n) = f(t_1, \dots, t_n).$$

Every variable-free tree $t \in T(\Sigma)$ has a value in every Σ-algebra $\mathbb{A}$. It is the value at t of the unique homomorphism $h : \mathbb{T}(\Sigma) \to \mathbb{A}$.

The existence of a unique homomorphism from the Σ-algebra of trees into an arbitrary Σ-algebra $\mathbb{A}$ provides also the basis for the view that regards the elements of $T(\Sigma, X_n)$ as *derived operations.* Each tree $t \in T(\Sigma, X_n)$ induces an n-ary function

$$t_{\mathbb{A}} : A^n \to A$$

The meaning of this function $t_{\mathbb{A}}$ is defined in the following way. For every $(a_1, \ldots, a_n) \in A^n$ $t_{\mathbb{A}}(a_1, \ldots, a_n) = \hat{a}(t)$, where $\hat{a} : \mathbb{T}(\Sigma, X_n) \to \mathbb{A}$ is the unique homomorphism with $\hat{a}(x_i) = a_i$.

In the particular case where $\mathbb{A}$ is the Σ-algebra $\mathbb{T}(\Sigma, X_m)$ of trees over Σ that contain at most variables from $X_m = \{x_1, \ldots, x_m\}$ at their leaves the unique homomorphism extending the assignment of a tree $t_i \in T(\Sigma, X_m)$ to the variable x_i in X_n acts as a substitution:

$$t_{\mathbb{T}(\Sigma, X_m)}(t_1, \ldots, t_n) = t[t_1, \ldots, t_n]$$

where the last tree indicates the result of substituting t_i for x_i in t.

Let V be an ordinary finite alphabet. It gives rise to a *monadic* signature Σ_V if all the members of V are assigned rank one and a new symbol ε is added as the single constant of rank zero. As was pointed out in the previous paragraph, there is a unique homomorphism from the trees over Σ_V, considered as the carrier of a Σ_V-algebra, to any other algebra of the same type. In particular, there is a homomorphism into V^* when every $a \in V$, i.e. $a \in (\Sigma_V)_1$, is interpreted as left-concatenation with the symbol a and when ε is interpreted as the constant string of length zero. This homomorphism is actually an isomorphism between $T(\Sigma_V)$ and V^* (cf. Maibaum, 1974). This isomorphism will play an important role in the next section on context-free string languages.

We now turn to introducing the notion of a context-free tree grammar (cftg). This type of grammar is related to the type of grammars that were defined by Fischer (1968) and were called macro grammars by him. Context-free tree grammars constitute an algebraic generalization of macro grammars since the use of macro-like productions served the purpose of making simultaneous string copying a primitive operation.

Definition 5. A **context–free tree grammar** $G = \langle \Sigma, \mathcal{F}, S, P \rangle$ is a 4-tuple, where Σ is a finite ranked alphabet of *terminals*, $\mathcal{F}$ is a finite ranked alphabet of *nonterminals* disjoint from Σ, $S \in \mathcal{F}$ is the start symbol of rank 0 and P is a finite set of rules of the form

$$F(x_1, \ldots, x_m) \to t \qquad (n \geq 0)$$

where $F \in \mathcal{F}_m$ and $t \in T(\Sigma \cup \mathcal{F}, X_m)$. Recall that X is here assumed to be a set of variables $X = \{x_1, x_2, \dots\}$ and $X_m = \{x_1, \dots, x_m\}$.

In the conventional case of context-free string grammars the class of languages generated by unrestricted derivations is identical to the class of languages generated by left-most derivations (or by right-most derivations, for that matter). Such an equivalence cannot be proved in the tree case. While this is true for cftg's in general, the difference between the derivation modes has no effect on the special category of monadic linear tree grammars, which will be our only concern in the sequel.

The following definition records the unrestricted derivation mode.

Definition 6. Let $G = \langle \Sigma, \mathcal{F}, S, P \rangle$ be a cftg and let $t, t' \in T(\Sigma \cup \mathcal{F})$. t' is **directly derivable** from t ($t \Rightarrow t'$) if there is a tree $t_0 \in T(\Sigma \cup \mathcal{F}, X_1)$ containing exactly *one* occurrence of x_1, a corresponding rule $F(x_1, \dots, x_m) \to t''$, and trees $t_1, \dots, t_m \in T(\Sigma)$ such that

$$\begin{array}{rcl} t & = & t_0[F(t_1, \dots, t_m)] \\ t' & = & t_0[t''[t_1, \dots, t_m]] \end{array}$$

t' is obtained from t by replacing an occurrence of a subtree $F(t_1, \dots, t_m)$ by the tree $t''[t_1, \dots, t_m]$.

Recall from above that for $m, n \geq 0$, $t \in T(\Sigma, X_m)$ and $t_1, \dots, t_m \in T(\Sigma, X_n)$ $t[t_1, \dots, t_m]$ denotes the result of **substituting** t_i for x_i in t. Observe that $t[t_1, \dots, t_m]$ is in $T(\Sigma, X_n)$.

As is customary $\overset{*}{\Rightarrow}$ denotes the transitive-reflexive closure of $\Rightarrow$.

Definition 7. Suppose $G = \langle \Sigma, \mathcal{F}, S, P \rangle$ is a cftg. We call

$$\mathcal{L}(G, S) = \{t \in T(\Sigma) \mid S \overset{*}{\Rightarrow} t\}$$

the **context-free tree language** generated by G from S.

We reserve a special definition for the case where $\mathcal{F}$ contains only function symbols of rank zero.

Definition 8. A **regular tree grammar** is a tuple $G = \langle \Sigma, \mathcal{F}, S, P \rangle$, where Σ is a finite ranked alphabet of terminals, $\mathcal{F}$ is a finite alphabet of function or nonterminal symbols of rank zero, $S \in \mathcal{F}$ is the start symbol and $P \subseteq \mathcal{F} \times T(\Sigma \cup \mathcal{F})$ is a finite set of productions. The **regular tree language** generated by G is

$$\mathcal{L} = \{t \in T(\Sigma) \mid S \overset{*}{\Rightarrow} t\}$$

Note that in the case of regular grammars the analogy with the conventional string theory goes through. There is an equivalence of the unrestricted, the rightmost and the leftmost derivation modes where the terms 'rightmost' and 'leftmost' are to be understood with respect to the linear order of the leaves forming the frontier of a tree in a derivation step.

3 Monadic Trees

Let us view grammars as a mechanism in which local transformations on trees can be performed in a precise way. The central ingredient of a grammar is a finite set of productions, where each production is a pair of trees. Such a set of productions determines a binary relation on trees such that two trees t and t' stand in that relation if t' is the result of removing in t an occurrence of a first component in a production pair and replacing it by the second component of the same pair. The simplest type of such a replacement is defined by a production that specifies the substitution of a single-node tree t_0 by another tree t_1. Two trees t and t' satisfy the relation determined by this simple production if the tree t' differs from the tree t in having a subtree t_1 that is rooted at an occurrence of a leaf node t_0 in t. In slightly different terminology, productions of this kind incorporate instructions to rewrite auxiliary variables as a complex symbol that, autonomously, stands for an element of a tree algebra. As long as the carrier of a tree algebra is made of constant tree terms the process whereby nullary variables are replaced by trees is analogous to what happens in string languages when a nonterminal auxiliary symbol is rewritten as a string of terminal and non-terminal symbols, independently of the context in which it occurs. The situation changes dramatically if the carrier of the algebra is made of symbolic counterparts of derived operations and the variables in production rules range over such second-level entities. The following example illustrates the gain in generative power to be expected from production systems determining relations among trees that derive from second-order substitution of operators rather than constants.

Example 2. Let us assume that we are dealing with a vocabulary V that contains the symbols a and b as its only members. Trees over the associated monadic signature $\Sigma = \Sigma_0 \cup \Sigma_1$, where $\Sigma_0 = \{\varepsilon\}$ and $\Sigma_1 = \{a, b\}$ are arbitrary sequences of applications of the operators a and b to the constant ε. As will be recalled from the preceding section, trees in $T(\Sigma, X)$ can be regarded as derived operators. Due to the fact that Σ is a *monadic* signature these trees may not contain more than a single variable. Auxiliary symbols ranging over this domain take therefore as values monadic or constant derived operators. When monadic auxiliary symbols appear in productions this means that they behave the way that nullary auxiliary symbols do except for the fact that their argument has to be inserted into the unique variable slot of their replacing derived operator. After these explanations it should be obvious that the following grammar over the *monadic* signature Σ generates the context-free string language $\{a^n b^n\}$ when combined with the unique isomorphism mentioned above:

$$G = \langle \Sigma, \mathcal{F}, S, P \rangle$$

$$\mathcal{F}_0 \;=\; \{s\} \qquad\qquad \mathcal{F}_1 \;=\; \{F\}$$

$$P = \left\{ \begin{array}{lcl} S & \to & F(\varepsilon) \,|\, \varepsilon, \\ F(x) & \to & a(F(b(x))) \,|\, a(b(x)) \end{array} \right\}$$

$$\mathcal{L}(G, S) = \{\overbrace{a(a\ldots}^{n}\overbrace{(b(b\ldots}^{n}(\varepsilon)\ldots)\}$$

The grammar in the last example illustrates for the language $L = \{a^n b^n\}$ a transformation that can be applied to the grammar of any given context-free language.

Claim 1. *For every finite alphabet V, the context-free tree subsets of $T(\Sigma_V)$ (where Σ_V is the monadic signature induced by V) have exactly the context-free string languages as values of the unique isomorphism that maps trees in $T(\Sigma_V)$ into strings in V^* and interprets the elements of V as unary operations that concatenate for the left with their argument.*

Proof. Let $G = (N, V, S, P)$ be an arbitrary context-free string grammar, where N is a finite set of nonterminal symbols, V the finite terminal alphabet disjoint from $N, S \in N$ the initial or start symbol and $P \subseteq Nx(N \cup V)*$ a finite set of productions. We associate with G an equivalent context-free tree grammar $G' = (\Sigma_V, \mathcal{F}, S, P')$ in the following way. We let Σ_V be the monadic ranked alphabet corresponding to V. We let $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ where $\mathcal{F}_0 = \{S'\}$ and $\mathcal{F}_1 = N$, be the ranked alphabet of nonterminals. The new set of productions P' is obtained as follows. If $A \to w_1 \ldots w_n$ is in P, where $w_i \in N \cup V$ then $A(x) \to w_1(w_2(\ldots(w_n(x)\ldots)$ is in P'. In addition, we have a rule for the new start symbol: $S' \to S(\varepsilon)$. It should be clear from the preceding example that $L(G) = h(L(G'))$ where h denotes the unique isomorphism between $V*$ and $T(\Sigma_V)$. For the other direction assume that $G = (\Sigma, \mathcal{F}, S, P)$ is an arbitrary cftg over the *monadic* terminal alphabet Σ. We associate with G an equivalent context-free string grammar $G' = (\mathcal{F}, \Sigma, S, P')$ in the following way. $\mathcal{F}$ and Σ are regarded as unranked sets and play the role of nonterminal and terminal alphabets, respectively. The set of productions P' is obatined from the paths in the members of P. More precisely. Let $F(x_1, \ldots, x_n) \to t$ be a member of P with $t \in T(\Sigma \cup \mathcal{F}, X_n)$. Then each t' is the concatenation of symbols from Σ and $\mathcal{F}$ that label a path through t. It should be obvious that nonterminals in $\mathcal{F}_n$ for $n > 1$ have to be ultimately replaced by linear or constant monadic trees in a derivation according to the original cftg G if they are to contribute to the generated language at all. Spurious nonterminals are retained in the process of substituting the path sets for the trees on the right hand sides of the rules in P, but they still do not affect the string languages specified by G'. Indeed, under the unique isomorphism between $V*$ and $T(\Sigma)$ it holds that G and G' generate the same language. $\square$

Let V be an ordinary finite alphabet and Σ_V ist associated monadic signature. rcall that all nontermianls in a regular tree grammar are of arity zero. An arbitrary rule in a regular tree grammar over the monadic signature Σ_V can therefore assume only one of the following two forms:

$$A \to a_1(a_2(\dots a_n(\varepsilon)\dots))$$
$$A \to a_1(a_2(\dots a_n(B)\dots))$$

where $a_i \in (\Sigma_V)_1$, i.e. the a_i's are the monadic terminal operators corresponding to the members of V and A and B are two nonterminals of rank zero. Under the unique isomorphism that relates strings with monadic "vertical" trees the two rule formats above correspond exactly to the two types of rules in regular string grammars. This implies that a language L generated by a regualr tree grammar over a unary ranked alphabet is, interpreted as a string language, the language generated by the corresponding regular string grammar. Based on the claim about the relationship between context-free string grammars and context-free tree grammars over a finite vocabulary and its associated monadic signature, respectively, we have therefore established the result, announced in the introduction that the increase in generative power which characterizes the shift from the regular to the context-free languages, is due to the transgression from a nullary, first-order substitution process to its higher-order analogue.

4 Adjoining Trees

Very early in the development of (regular) tree grammars it was realized that there exists a close relationship between the families of trees generated by tree grammars and the family of context-free string languages. This fundamental fact is best described by looking at it from the perspective on trees that views them as symbolic representations of values in arbitrary domains.

Definition 9. Suppose Σ is a ranked alphabet. We call **yield** or **frontier** the unique homomorphism y that interprets every operator in Σ_n as the n-ary operation of concatenation. More precisely

$$\begin{array}{rcll} y(f) & = & f & \text{for } f \in \Sigma_0 \\ y(f(t_1,\dots,t_n)) & = & y(t_1)\dots y(t_n) & \text{for } f \in \Sigma_n \text{ and } t_i \in T(\Sigma) \end{array}$$

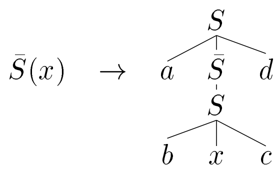
Fact *A (string) language is context-free iff it is the yield of a regular tree language.*

As was shown in the last section, the addition of macro variables or n-ary nonterminals ($n > 0$) increases the generative power of cftg's over monadic alphabets considerably. In addition, the transformation of an arbitrary context-free tree grammar over a monadic alphabet into an equivalent context-free string grammar has shown that it is only the monadic nonterminals that are operative in the derivation process with a terminal outcome, i.e. a (tree-)expression over the terminal alphabet. The obvious question that presents itself in this situation is whether the addition of monadic nonterminals to a regular grammar over an arbitrary signature leads to a langugae family that has already been introduced for independent reasons.

Example 3. Let $G = (\Sigma, \mathcal{F}, S', P)$ be a context-free tree grammar with the following specifications:

$$\Sigma = \Sigma_0 \cup \Sigma_3 \text{ where } \Sigma_0 = \{a, b, c, d\} \text{ and } \Sigma_3 = \{S\}$$
$$\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \text{ where } \mathcal{F}_0 = \{S'\} \text{ and } \mathcal{F}_1 = \{\bar{S}\}$$
$$P = \{S' \to \bar{S}(\varepsilon), \quad \bar{S}(x) \to x, \quad \bar{S}(x) \to S(a, \bar{S}(S(b, x, c)), d)\}$$

In tree form the last rule has the following shape:



This grammar generates the string language $\{a^n b^n c^n d^n\}$, i.e., $y(L(G)) = \{a^n b^n c^n d^n\}$.

Apart from minor notational modifications the grammar in the last example corresponds to a tree adjoining grammar which serves as an illustration in the paper by Vijay-Shanker and Weir (1992). Note that apart from the start symbol the only other nonterminal is of arity one. As was the case in connection with the context-free string languages, the preceding example is a particular instance of the general situation. The tree adjoining languages correspond to the context-free tree languages that are specified by rule systems with nonterminals of arity at most one.

We shall prove the equivalence between tree adjoining languages and context-free tree languages with a monadic rule system for the case of tree adjoining grammars with a restricted set of adjunction constraints.

The device of *adjunction constraints* was introduced by A. Joshi for the purpose of specifying which auxiliary trees can be adjoined at a given node. The distinction in cftg's between functional nonterminal and terminal labels corresponds to the distinction between nonterminals with an obligatory and a null adjunction constraint in an initial or an auxiliary tree of a tree adjoining grammar. In the theorem below we shall treat this particular distinction. The proof can be easily adapted to the general case.

Definition 10. A **tree adjoining grammar** (tag) G is a quintuple $G = (V, N, S, \mathbb{I}, \mathbb{A})$, where (i) V is a set of terminal symbols, (ii) N is a finite set of nonterminal symbols disjoint from V, (iii) $S \in N$ is the distinguished start symbol, (iv) $\mathbb{I}$ is a finite set of trees, called **initial trees**, whose root label is S and (v) $\mathbb{A}$ is a finite set of trees, called **auxiliary trees**, that have a distinguished frontier node, called **foot node**, whose label is identical to the label of the root node. Only frontier nodes in $\mathbb{I}$ or $\mathbb{A}$ are labelled by terminal symbols. Nonterminal symbols can label both frontier and interior nodes. The case where a node is specified by an obligatory adjunction constraint is indicated by a bar over the node label.

Definition 11. Let $G = (V, N, S, \mathbb{I}, \mathbb{A})$ be a tag and let t, t' be two trees with node labels from V and N. t' is **directly derivable** from t ($t \Rightarrow t'$) if there is a tree t_0 conatining exactly *one* occurrence of a new leaf label ζ, an auxiliary tree t_A with root and foot node label A such that

$$t = t_0(t''/\zeta) t' = \qquad t_0(t_A(t''/A)/\zeta)$$

where the root of t'' is labelled by $\bar{A}$ and where the notation $t_1(t_2/X)$ indicates the result of substituting the tree t_2 for a particular leaf node of t_1 which is labelled by X.

Definition 12. Suppose $G = (V, N, S, \mathbb{I}, \mathbb{A})$ is a tag. The **tree language** $T(G)$ generated by G is defined as

$$T(G) = \{t | t_1 \overset{*}{\Rightarrow} t \text{ for some } t_i \in \mathbb{I} \text{ and } t \text{ has no obligatory node labels}\}$$

The *string language*, $L(G)$, generated by G is defined as

$$L(G) = \{\mathit{frontier}(t) | t \in T(G) \text{ and } \mathit{frontier}(t) \in V*\}$$

The following theorem shows how to construct a weakly equivalent tag G' for a given cftg G with contains only nullary and monadic nonterminals and vice versa.

Theorem 2. *The classes of string languages generated by tree adjoining grammars and by cftg's with nonterminals of arity at most one and linear production rules are the same.*

Remark 1. The linearity constraint mirrors the requirement of a distinguished foor node in the auxiliary trees.

Proof. Let $G = (V, N, S, \mathbb{I}, \mathbb{A})$ be a tag. We construct an equivalent cftg $G' = (\Sigma, \mathcal{F}, S', P)$, where we let $\Sigma_0 = V$ and $\Sigma_n = \{A_n | A \in N$ and A labels a node in $\mathbb{I}$ or $\mathbb{A}$ with n daughters$\}$, $\mathcal{F}_0 = \{A' | A \in N$ and $\mathcal{F}_1 = \{\bar{A} | A \in N\}$ and we let P consist of the following productions:

$$\begin{aligned}
S' &\to t_i^* \text{ for } t_i \in \mathbb{I} \\
A' &\to t_a^* \text{ for } t_a \in \mathbb{A} \text{ with } A \text{ labelling the root node} \\
\bar{A}(x) &\to t_a^*(x/A) \text{ for } t_a \in \mathbb{A} \text{ with } A \text{ labelling root and foot node, and} \\
& \quad A \in N \text{ is annotated with the null adjunction constraint} \\
\bar{A}(x) &\to t_a^*(\bar{A}(x)/\bar{A}) \text{ otherwise}
\end{aligned}$$

The trees t^* are the result of replacing every internal node with a nonterminal label A cattying the obligatory adjunction constraint by the branch consisting of the node labelled $\bar{A}$ immediately dominating the node labelled A and of replacing every nonterminal non-foot leaf label A with the same constraint by A'. In the construction of the rule set P we have suppressed the numerical subscripts of the terminal symbols originating from the alphabet N. A straightforward, but tedious inductive argument will show that G' is weakly equivalent with G and even generates the same tree language.

Suppose now that $G = (\Sigma, \mathcal{F}, S, P)$ is a cftg satisfying the constraints in the statement of the theorem. An equivalent tag $G' = (V, N, S', \mathbb{I}, \mathbb{A})$ is obtained in the following manner. Let $V = \Sigma_0$ and let $N = \bigcup_{n>0} \Sigma_n \cup \mathcal{F} \times P$. □

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